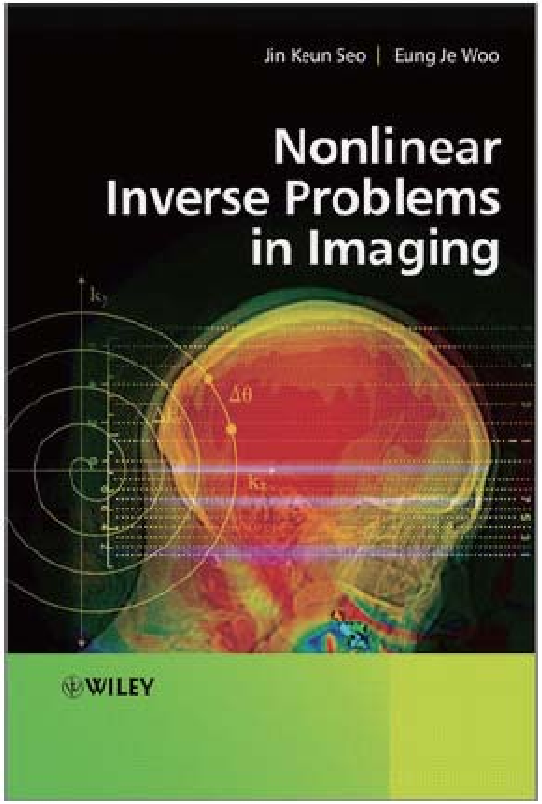
Partial Differential Equations



Part IV: Euler-Lagrange Equations

Homepage

http://www.seojinkeun.com/♯!pde-course-/c1laj

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Optimization problems

The optimization problem is to find the best solution from all possible solutions. We want to minimize an objective function $J(u)$ (the energy, cost, entropy, performance, and so on) within a set W.

$$\boldsymbol{Minimize\ J(u)}$$

$$within\ \ the\ set\ \ W$$

If it is difficult to find the minimizer analytically, we use iteration method

$$\mathbf{u^{n+1}} = \mathbf{u^{n}} - \lambda \nabla J(u^{n})$$

$$\nabla = ??$$

The Euler-Lagrange equation of $J(u)$ is somewhat $\nabla J(u) = 0$, where ∂ is "Freschet" derivative.

Minimization in Finite Dimension

Minimization

Assume $\mathbf{u}_* \in W = \mathbb{R}^n$ is a global minimum of the objective function $J(\mathbf{u})$ on the domain W, that is,

$$J(\mathbf{u}_*) \leq J(\mathbf{v}) \qquad \text{for all} \quad \mathbf{v} \in W.$$

Then $\mathbf{u}_*$ is a critical point of J:

$$0 = \frac{d}{dt} J(\mathbf{u}_* + t\mathbf{v})|_{t=0} = \nabla J(\mathbf{u}_*) \cdot \mathbf{v} = \langle \nabla J(\mathbf{u}_*), \mathbf{v} \rangle \qquad \text{for all } \mathbf{v} \in \mathbb{R}^n.$$

The left side of the above identity is known as the directional derivative of J with respect to $\mathbf{v}$ that is defined as

$$\langle \nabla J(\mathbf{u}_*), \mathbf{v} \rangle = \sum_{j=1}^{n} \partial_j J(\mathbf{u}_*)\, v_j.$$

Note that $-\nabla J(u)$ is the steepest descent direction of J at u.

Minimization in Sobolev space $H^1(\Omega)$

$$\textit{Minimize } J(u)$$

$$\textit{within the set } W := \{ v \in H^1(\Omega) : v|_{\partial\Omega} = f \}$$

u_* is minimizer iff $J(u_*) \le J(v) \quad \forall v \in W$

iff $J(u_*) \le J(u + t\phi) \quad \forall \phi \in H_0^1(\Omega), \ \forall t \in R$

$$\Rightarrow \quad \frac{d}{dt} J(u + t\phi)\,|_{t=0} = 0 \qquad \forall \phi \in H_0^1(\Omega)$$

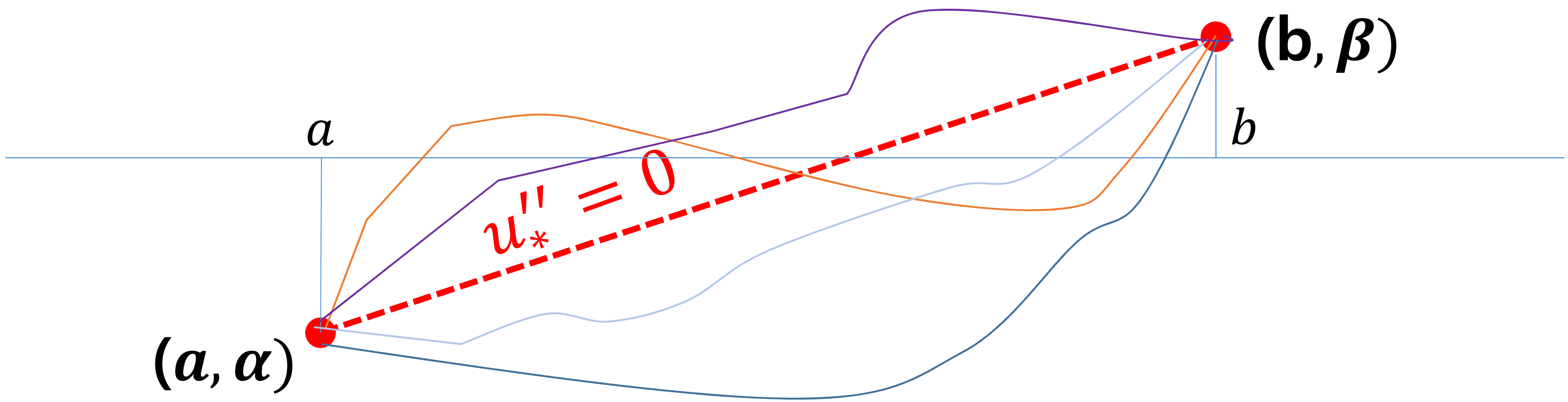
Length Minimization joining two points

Consider the curve minimization problem joining two point (a, α) and (b, β):

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) = \int_a^b \sqrt{1 + (u')^2}\, dx \\ \text{within the set} & W = \{u \in W^{1,1}(a, b) \mid u(a) = \alpha, u(b) = \beta\} \end{array} \right.$$

Here, $W^{1,1}(a, b) = \{u \mid \int_a^b \sqrt{u^2 + (u')^2}\, dx < \infty\}$. (This is a kind of Sobolev space or a Banach space.) If u_* is a minimizer, then

$$u_*''(x) = 0 \qquad a < x < b.$$



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$$u_*''(x) = 0 \qquad a < x < b.$$

Proof. Assume that u is minimizer

- For all $v \in H_0^1$, we have

$$\begin{aligned} \frac{d}{dt}J(u+tv)|_{t=0} &= \frac{d}{dt}\int_a^b \sqrt{1+(u'+tv')^2}dx\Big|_{t=0} \\ &= \int_a^b \frac{2u'v'+2tv'^2}{\sqrt{1+(u'+tv')^2}}dx\Big|_{t=0} = \int_a^b \frac{2u'v'}{\sqrt{1+(u')^2}}dx \\ &= 2\int_a^b \frac{d}{dx}\left[\frac{u'}{\sqrt{1+(u')^2}}\right] vdx \\ &= 2\int_a^b \frac{u''[\sqrt{1+(u')^2}-2u'^2]}{1+(u')^2}\ v\ dx \end{aligned}$$

Consider the curve minimization problem joining two point (a, α) and (b, β):

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) = \int_a^b \sqrt{1+(u')^2}dx \\ \text{within the set} & W = \{u \in W^{1,1}(a,b) \mid u(a) = \alpha, u(b) = \beta\} \end{array} \right.$$

Here, $W^{1,1}(a,b) = \{u \mid \int_a^b \sqrt{u^2+(u')^2}dx < \infty\}$. (This is a kind of Sobolev space or a Banach space.) If u_* is a minimizer, then

$$u_*''(x) = 0 \qquad a < x < b.$$

$$\frac{d}{dt} J(u+tv)|_{t=0} \quad = 2\int_a^b \frac{u''[\sqrt{1+(u')^2} - 2u'^2]}{1+(u')^2} v\, dx$$

- Since the above identity holds for all $v \in H_0^1$, the minimizer u_* satisfies Euler-Lagrange

$$\frac{u_*''[\sqrt{1+(u_*')^2} - 2u_*'^2]}{1+(u_*')^2} = 0 \qquad a < x < b$$

and hence

$$u_*''(x) = 0 \qquad a < x < b.$$

Hence, the straight line joining (a, α) and (b, β) is the minimizer.

Dirichelt Problem

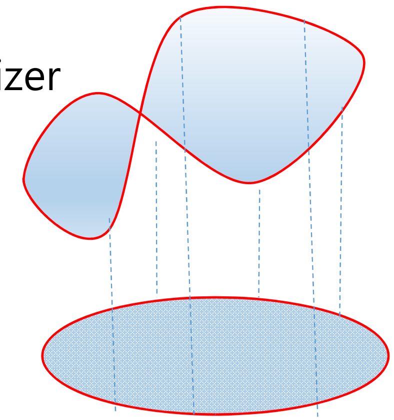
Let Ω be a domain in $\mathbb{R}^2$. Consider the minimization problem

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega |\nabla u(x,y)|^2 dxdy \\ \text{within the set} & W := \{u \in H^1(\Omega) : u|_{\partial\Omega} = f(x,y)\} \end{array} \right.$$

Here, $H^1(\Omega) = \{u : \int_\Omega |u|^2 + |\nabla u|^2 dxdy < \infty\}$. Then the minimizer u satisfies the Dirichlet problem:

$$\begin{cases} \nabla^2 u(x,y) = 0 & (x,y) \in \Omega \\ u|_{\partial\Omega} = f & (\text{ the prescribed boundary potential}) \end{cases}$$

Proof. Assume that u is minimizer



- For all $v \in H_0^1(\Omega) := \{v \in H^1(\Omega) \ : \ u|_{\partial\Omega} = 0\}$,

$$\begin{aligned} 0 &= \tfrac{d}{dt} J(u + tv)|_{t=0} = 2\int_\Omega \nabla u \cdot \nabla v \ dxdy \\ &= -2\int_\Omega \nabla^2 uv \ dxdy \end{aligned}$$

Since this hold for all $v \in H_0^1(\Omega)$, we have

$$0 = \nabla^2 u \qquad \text{in } \Omega$$

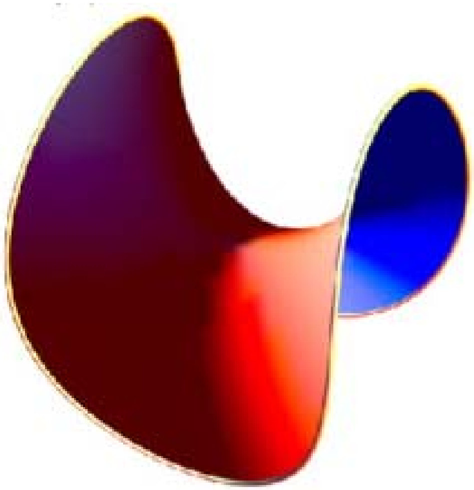
Minimal Surfaces

Consider the minimal surface problem

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega \sqrt{1 + u_x^2 + u_y^2}\,dxdy \\ \text{within the set} & W := \{u \in W^{1,1}(\Omega) : u|_{\partial\Omega} = f\} \end{array} \right.$$

Here, $W^{1,1}(\Omega) = \{u : \int_\Omega |u| + |\nabla u| dxdy < \infty\}$. If a minimizer u exist, u satisfies the following nonlinear problem with the Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = f & (\text{ Dirichlet boundary}) \end{cases}$$



- For all $v \in W_0^{1,1}(\Omega) := \{v \in W^{1,1}(\Omega) : v|_{\partial\Omega} = 0\}$,

$$\begin{aligned} 0 &= \frac{d}{dt} J(u + tv)|_{t=0} = \int_\Omega \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla v \; dxdy \\ &= \int_\Omega \left[-\nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) \right] v \; dxdy \end{aligned}$$

Since this hold for all $v \in W_0^{1,1}(\Omega)$, we have

$$0 = \nabla \cdot (\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) \qquad \text{in } \Omega$$

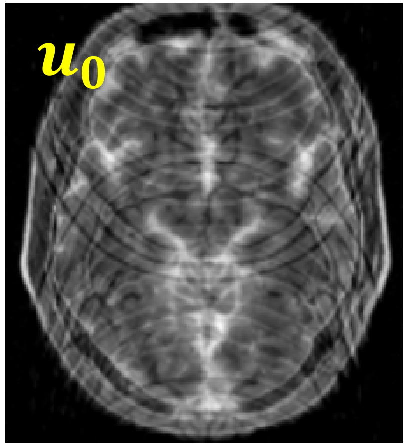
Total Variation Minimization

Consider the minimization problem

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega |\nabla u(x,y)| + |u-u_0|^2 dxdy \\ \text{within the set} & W := \{u \in W^{1,1}(\Omega) : \mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0\} \end{array} \right.$$

If a minimizer u exist, u satisfies the following nonlinear problem with zero Neumann boundary condition:

$$\begin{cases} \nabla \cdot (\frac{\nabla u}{|\nabla u|}) = 2(u-u_0) & \text{in} \quad \Omega \\ \mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0 & (\text{ insulating boundary}) \end{cases}$$

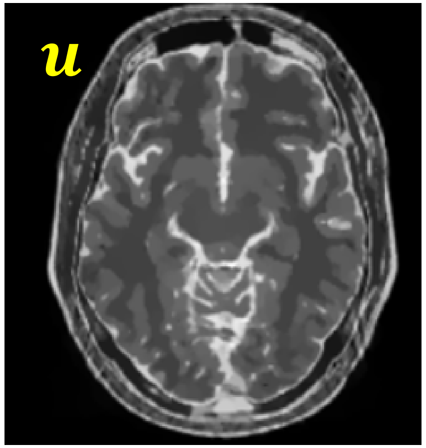


- For all $v \in W^{1,1}(\Omega)$,

$$\begin{aligned} 0 &= \frac{d}{dt} J(u+tv)|_{t=0} = \int_\Omega \frac{\nabla u}{|\nabla u|} \cdot \nabla v + 2(u-u_0)v \, dxdy \\ &= \int_\Omega \left[-\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + 2(u-u_0) \right] \, v \, dxdy \end{aligned}$$

Since this hold for all $v \in W^{1,1}(\Omega)$, we have

$$0 = -\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + 2(u-u_0) \qquad \text{in } \Omega$$

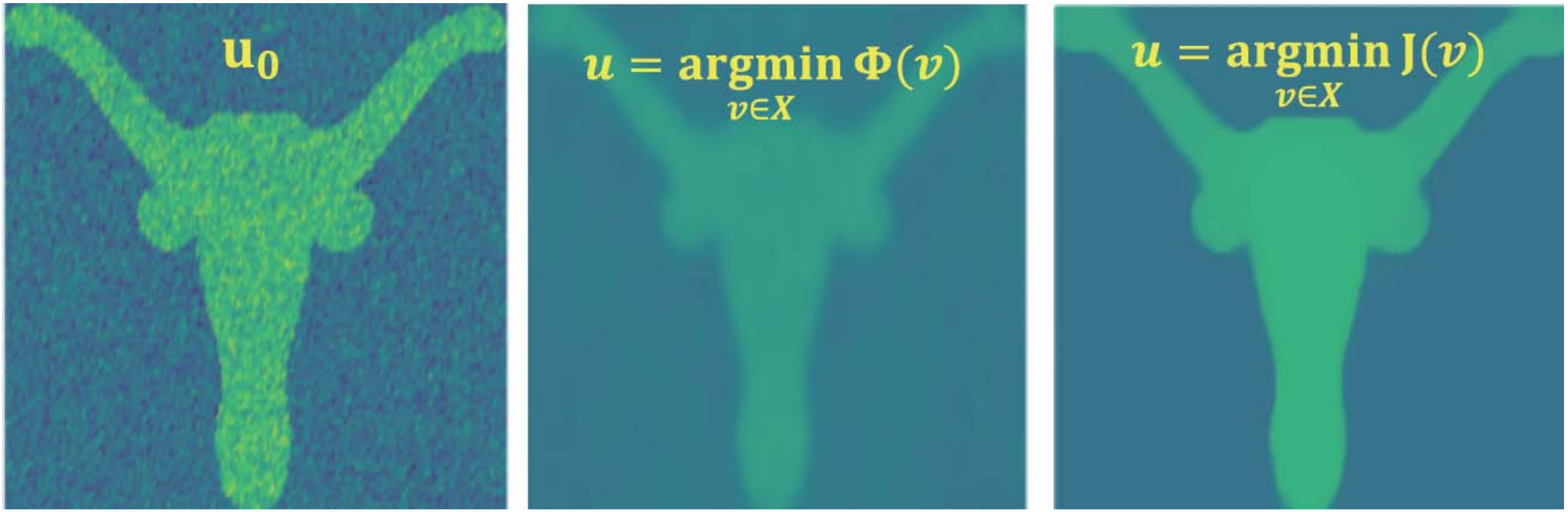


Comparison of L^2 & $L^1 - norm$ minimization

Let $u_0 \in H_0^1(\Omega)$ represent a noisy image defined in Ω. Consider the two minimization problems.

$$Minimize\ \ \Phi(u) := \int_\Omega |\nabla u|^2 + |u-u_0|^2\,dx$$
$$within\ \ the\ set\ \ X := H_0^1(\Omega)$$

$$Minimize\ \ E(u) := \int_\Omega |\nabla u|^1 + |u-u_0|^2\,dx$$
$$within\ \ the\ set\ \ X \coloneqq W_0^{1,1}(\Omega).$$



$$\frac{d}{dt}\Phi(u+t\phi)\Big|_{t=0} = -2\int_{\Omega}\left(\nabla^2 u - u + u_0\right)\phi \qquad \forall\phi \in H_0^1(\Omega)$$

- Hence, $\left(\nabla^2 u - u + u_0\right)$ is the steepest descent direction of Φ at u, the direction that $\Phi(u)$ decreases most quickly. Explain the reason.

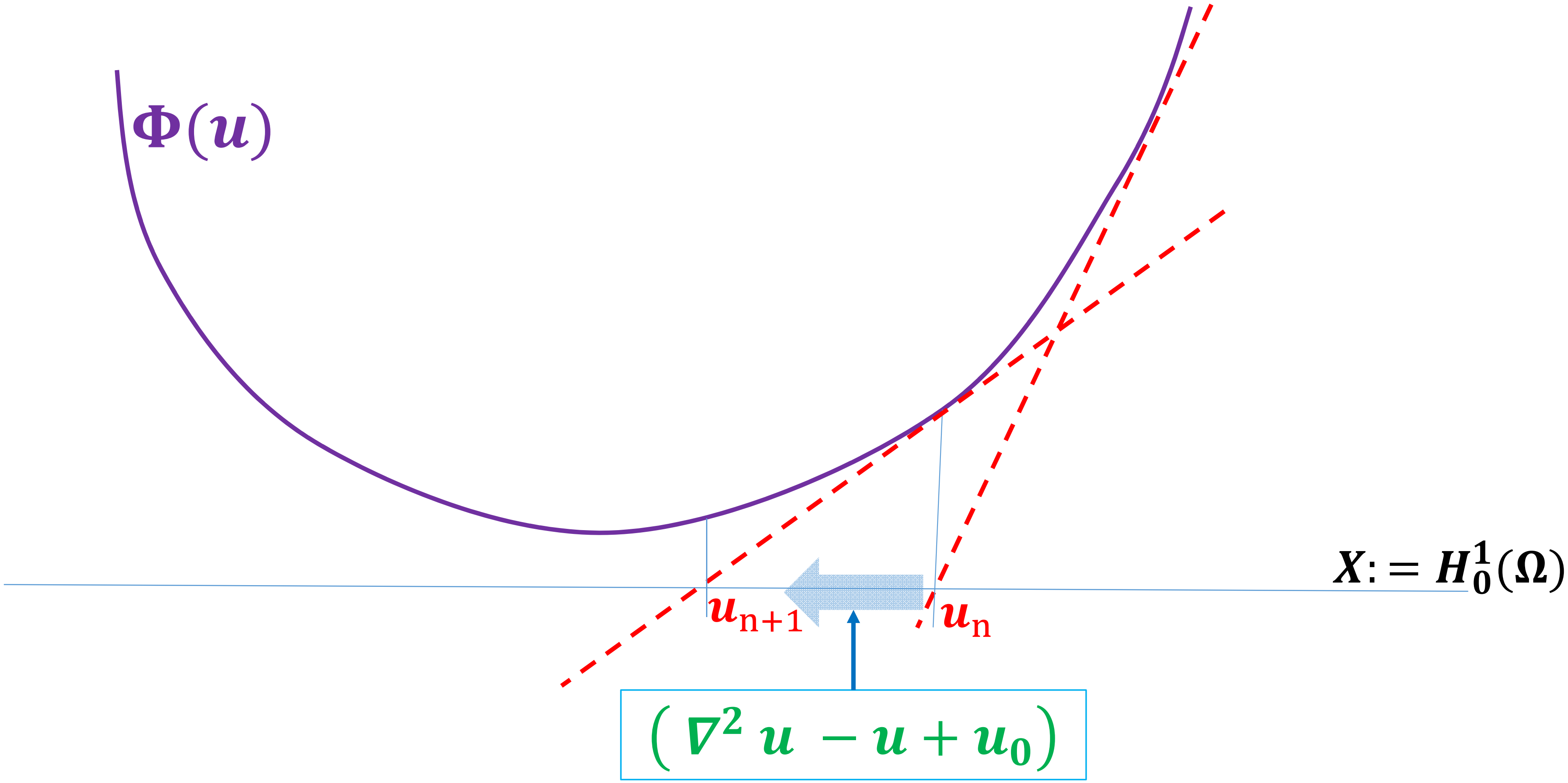
- The corresponding iteration schemes is

$$u_{n+1} = u_n - \lambda \nabla \Phi(u_n)$$

$$\frac{\partial u}{\partial t} = \lambda\left(\nabla^2 u - u + u_0\right) \quad in\ \Omega, \qquad t > 0$$

$$\textit{Minimize}\ \ \boldsymbol{\Phi}(\boldsymbol{u}) := \int_{\Omega} |\nabla \boldsymbol{u}|^2 + |u - u_0|^2\, dx$$

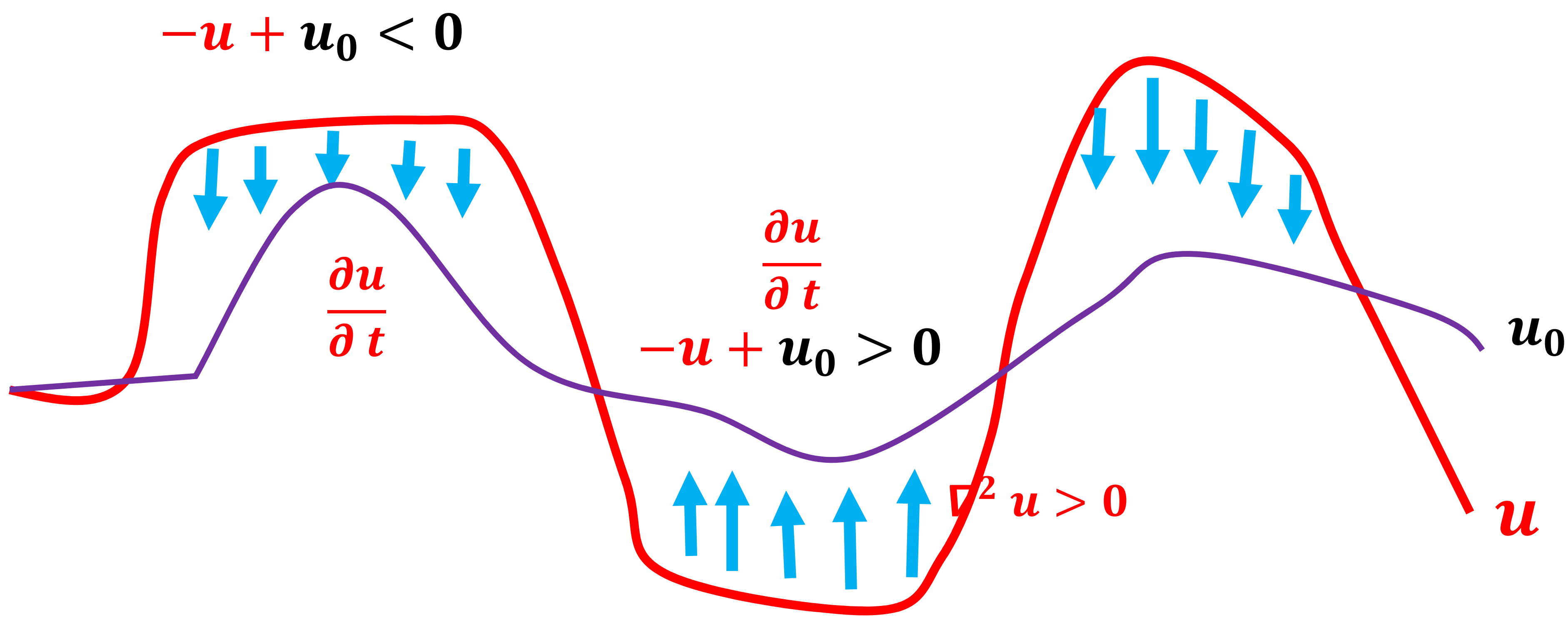
$$\textit{within the set}\ \ X := H_0^1(\Omega)$$



$$\frac{\partial u}{\partial t} = \lambda \left(\nabla^2 u - u + u_0 \right)$$

Fidelity Fitting

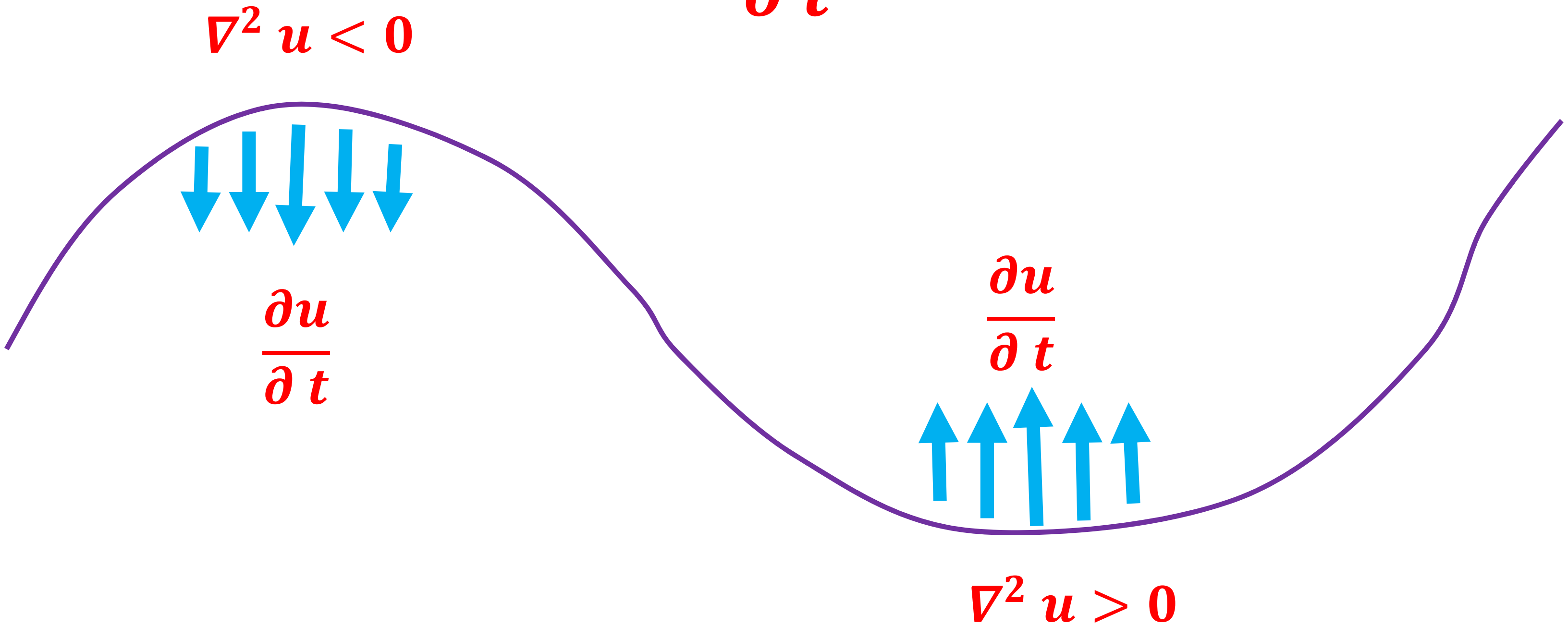
About $\frac{\partial u}{\partial t} = -u + u_0$ for image



$$\frac{\partial u}{\partial t} = \lambda\left(\nabla^2 u - u + u_0 \right)$$

Regularization

About $\frac{\partial u}{\partial t} = \nabla^2 u$ for image



$$\frac{d}{dt}E(u+t\phi)\Big|_{t=0} = -\int_\Omega \left(\nabla\cdot\left(\frac{1}{|\nabla u|}\nabla u\right) - 2(u-u_0)\right)\phi$$

- Hence, $\left(\nabla\cdot\left(\frac{1}{|\nabla u|}\nabla u\right) - 2(u-u_0)\right)$ is the steepest descent direction of E at u, the direction that $\mathrm{E}(u)$ decreases most quickly. Explain the reason.

- The corresponding iteration schemes is

$$u_{n+1} = u_n - \lambda \nabla \mathrm{E}\,(u_n)$$

$$\frac{\partial u}{\partial t} = \lambda\left(\nabla\cdot\left(\frac{1}{|\nabla u|}\nabla u\right) - 2(u-u_0)\right) \qquad in\ \Omega, \qquad t>0$$

Final Remark

During 1990-2015, PDE-based algorithms had a significant impact on many image processing tasks. Currently, image processing is experiencing a paradigm shift due to a marked and rapid advance in deep learning techniques. Deep learning techniques are being applied for various purposes in image analysis area.