Riemann integrable

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $f : \Omega \to \mathbb{R}$ be a bounded function.

- We enclose Ω in some rectangle $B = [a_1, b_1] \times [a_2, b_2]$ and extend f to the whole rectangle by defining it to be zero outside of Ω.

- Let $\mathcal{P}$ be a partition of B obtained by dividing $a_1 = x_0 < x_1 < \cdots < x_n = b_1$ and $a_2 = y_0 < y_1 < \cdots < y_m = b_2$:

$$\mathcal{P} = \{\underbrace{[x_i, x_{i+1}] \times [y_j, y_{j+1}]}_{=\text{subrectangle } R} \ : \ i = 0, 1, \cdots, n-1, \ j = 0, 1, \cdots, m-1\}.$$

- Define the upper sum of f:

$$U(f, \mathcal{P}) := \sum_{R \in \mathcal{P}} \sup\{f(x, y) \mid (x, y) \in R\} \times (\text{volume of } R)$$

- Define the lower sum of f:

$$L(f,\mathcal{P}) := \sum_{R\in\mathcal{P}} \inf\{f(x,y) \mid (x,y)\in R\} \times (\text{volume of } R)$$

- Define the upper integral of f on Ω by

$$\overline{\int_\Omega} f = \inf\{U(f,\mathcal{P}) \ : \ \mathcal{P} \text{ is a partition of } B\}$$

and the lower integral of f on Ω by

$$\underline{\int_\Omega} f = \sup\{L(f,\mathcal{P}) \ : \ \mathcal{P} \text{ is a partition of } B\}$$

- We say that f is **Riemann integrable or integrable** if

$$\overline{\int_\Omega} f = \underline{\int_\Omega} f.$$

- If f is **integrable** on Ω, we denote

$$\int_\Omega f = \overline{\int_\Omega} f = \underline{\int_\Omega} f.$$

Theorem. (Taylor's Theorem for the case $f \in C^3(\mathbb{R}^n)$)

Let $f : \mathbb{R}^3 \to \mathbb{R}$ *is of class* C^3. *For* $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$,
$\exists \mathbf{c} = \mathbf{x} + t_0\mathbf{h}, \quad 0 < t_0 < 1,$ *such that*

$$f(\mathbf{x}+\mathbf{h}) = \quad f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x}) h_i + \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}) h_i h_j$$

$$+\frac{1}{3!} \sum_{i,j,k=1}^{n} \left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\mathbf{x} + t_0\mathbf{h}) h_i h_j h_k \right)$$

$$\begin{aligned} f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) &= \int_0^1 \tfrac{d}{dt} f(\mathbf{x}+t\mathbf{h})dt = \int_0^1 \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(\mathbf{x}+t\mathbf{h})h_i dt \\ &= \sum_{i=1}^n \int_0^1 \tfrac{\partial f}{\partial x_i}(\mathbf{x}+t\mathbf{h})h_i \tfrac{d(t-1)}{dt} dt \qquad \left(\text{Why? } \tfrac{d(t-1)}{dt} = 1\right) \\ &= \sum_{i=1}^n \left[\tfrac{\partial f}{\partial x_i}(\mathbf{x})h_i - \int_0^1 \tfrac{d}{dt}\left(\tfrac{\partial f}{\partial x_i}(\mathbf{x}+t\mathbf{h})h_i \right) (t-1)\ dt \right] \\ &= \sum_{i=1}^n \tfrac{\partial f}{\partial x_i}(\mathbf{x})h_i + R_1(\mathbf{h},\mathbf{x}) \end{aligned}$$

where $R_1(\mathbf{h},\mathbf{x}) = \sum_{i,j=1}^n \int_0^1 (1-t) \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}+t\mathbf{h})h_i h_j \right)\ dt$

Using $\frac{d}{dt}\left(-\frac{(t-1)^2}{2!}\right) = (1-t)$ and integration by part,

$$\begin{aligned} R_1(\mathbf{h},\mathbf{x}) &= \textstyle\sum_{i,j=1}^{n} \int_0^1 \frac{d}{dt}\left(-\frac{(t-1)^2}{2!}\right)\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}+t\mathbf{h})h_i h_j\right)\, dt \\ &= \frac{1}{2!}\textstyle\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x})h_i h_j + R_2(\mathbf{h},\mathbf{x}) \end{aligned}$$

where

$$R_2(\mathbf{h},\mathbf{x}) := \sum_{i,j,k=1}^{n} \int_0^1 \frac{(t-1)^2}{2!}\left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\mathbf{x}+t\mathbf{h})h_i h_j h_k\right)\, dt$$

Recall the second mean value theorem for integral

$$\int_0^1 f(t)g(t)dt = g(t_0)\int_0^1 f(t)dt \quad \text{for some } 0 < t_0 < 1.$$

Hence, $\exists t_0, 0 < t_0 < 1$ such that

$$R_2(\mathbf{h},\mathbf{x}) = \sum_{i,j,k=1}^{n}\left(\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\mathbf{x}+t_0\mathbf{h})h_i h_j h_k\right)\underbrace{\int_0^1 \frac{(t-1)^2}{2!}dt}_{\frac{1}{3!}}.$$

Theorem. (Divergence Theorem)

Let Ω be a bounded smooth domain in R^3. The volume integral of the divergence of a C^1-vector field $\mathbf{F} = (F_1, F_2, F_3)$ equals the total outward flux of the vector $\mathbf{F}$ through the boundary of Ω;

$$\int_\Omega div\mathbf{F}(y)dy = \int_{\partial\Omega} \mathbf{F}(y) \cdot \mathbf{n}(y) dS_y.$$

where $\mathbf{n}$ is the unit outward normal vector on the boundary.

Proof. Assume that $\mathbf{F} \in [C^1(\mathbb{R}^3)]^3$ for simplicity.

- **Key idea.** From $\int_a^b f'(x)dx = f(b) - f(a)$, it is easy to show that the above identity is true if Ω can be decomposed of cubes.
- Hence, the above identity is true if the domain Ω is an union of cubes.
- Since the general smooth domain Ω can be viewed as a limit of the union of cubes, this completes the proof.

Theorem. (Stokes's Theorem)

Let S be a open smooth surface S with the boundary as a smooth contour C. The surface integral of the curl of a C^1-vector field $\mathbf{F}$ *over the surface S is equal to the closed line integral of the vector* $\mathbf{F}$ *along the contour C.*

$$\int_S Curl\mathbf{F} \cdot \mathbf{n}(y) dS_y = \oint_C F(y) \cdot dl_y.$$

Proof. Assume that $\mathbf{F} \in [C^1(\mathbb{R}^3)]^3$ for simplicity.

- **Key idea.** From $\int_a^b f'(x)dx = f(b) - f(a)$, it is easy to show that the above identity is true provided S is a rectangle.
- Hence, the above identity is true if the surface S can be decomposed of rectangles.
- Since the general surface S can be viewed as a limit of piecewise planer surfaces, this completes the proof.

Pointwise and uniform convergence

Definition. The sequence of functions $f_k \in C([0,1])$ is said to converges pointwise to f if for each $x \in [0,1]$, $f_k(x) \to f(x)$. We say that the sequence of functions $f_k \in C([0,1])$ converges uniformly to f if $\sup_{x\in[0,1]} |f_k(x) - f(x)| \to 0$.

- Let $f_k(x) = x^k$. Then f_k converges pointwise to $f(x) = \begin{matrix} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{matrix}$. But $\sup_{x\in[0,1]} |f_k(x) - f(x)| = 1$ for all k. Hence, the convergence is not uniform.
- Let $f_n(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$. Then f_n converges uniformly to $\sin x$ in $[0,1]$.

Inner Product space

Definition. Let V be a complex vector space. An **inner product** on V is a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ with the following properties :

1. $\langle \alpha f + \beta g\, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$ for all $f, g, h \in V$ and $\alpha, \beta \in \mathbb{C}$.
2. $\langle f, g \rangle = \overline{\langle g, f \rangle}$
3. $\langle f, f \rangle \geq 0$, and $\langle f, f \rangle = 0 \;\Rightarrow\; f = 0$

Theorem The space V of the continuous functions $f : [a, b] \to \mathbb{C}$ forms an inner product space if we define

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx.$$

Inner Product in $\mathbb{R}^n$.

- For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define inner product and norm:
  $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{n} x(j)y(j), \quad \|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$
- $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is said to be an orthonormal basis of $\mathbb{R}^n$ if
  1. $\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$
  2. $\|\mathbf{e}_j\| = 1,\ j = 1, \cdots, n$
  3. $\langle \mathbf{e}_j, \mathbf{e}_i \rangle = 0$ if $i \neq j$.
- For example, $\mathbf{e}_1 = (1, 0, \cdots, 0), \mathbf{e}_2 = (0, 1, 0, \cdots, 0), \ldots$.
- If $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is an orthonormal basis, then every $x \in \mathbb{R}^n$ can be represented uniquely by $\mathbf{x} = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{e}_j \rangle \mathbf{e}_j$
- If $V_m = span\{\mathbf{e}_1, \cdots, \mathbf{e}_m\}$, the element in V_m closest to $\mathbf{x}$ is $\mathbf{x}_m = \sum_{j=1}^{m} \langle \mathbf{x}, \mathbf{e}_j \rangle \mathbf{e}_j$ with $\|\mathbf{x} - \mathbf{x}_m\| = \sqrt{\sum_{j=m}^{n} \langle \mathbf{x}, \mathbf{e}_j \rangle^2}$.

This useful dot product properties in Euclidean space can be generalized to infinite dimensional spaces by introducing Hilbert space.

Inner Product space $V = C[a, b]$

Consider the space V of the continuous functions $f : [a, b] \to \mathbb{C}$ with the inner product $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$.

- Define the norm of f by $\|f\| = \sqrt{\langle f, f \rangle}$.
- Define the distance between f and g by $d(f, g) = \|f - g\|$.

For $f, g, h \in V$, we have

- **Cauchy-Schwarz inequality.** $|\langle f, g \rangle| \leq \|f\|\|g\|$
- **Minkowski inequality.** $\|f + g\| \leq \|f\| + \|g\|$
- **Parallelogram law.** $\|f + g\|^2 + \|f - g\|^2 = 2\|f\|^2 + 2\|g\|^2$
- **Pythagorean Theorem.**

$$\text{If } \langle f, g \rangle = 0, \quad \text{then} \quad \|f + g\|^2 = \|f\|^2 + \|g\|^2$$

Theorem. (Cauchy-Schwarz inequality)

If $\langle \cdot, \cdot \rangle$ is an inner product in a real vector space $\mathcal{V}$, then $|\langle f, g \rangle| \leq \|f\| \|g\|$

Proof.

- Suppose $g \neq 0$. Let $h = \frac{g}{\|g\|}$. It suffices to prove that $|\langle f, h \rangle| \leq \|f\|$. (Why? $|\langle f, g \rangle| \leq \|f\| \|g\|$ iff $|\langle f, h \rangle| \leq \|f\|$.)
- Denote $\alpha = \langle f, h \rangle$. Then

$$\begin{aligned} 0 &\leq \|f - \alpha h\|^2 = \langle f - \alpha h, f - \alpha h \rangle \\ &= \|f\|^2 - \alpha \langle h, f \rangle - \alpha \langle f, h \rangle + |\alpha|^2 \\ &= \|f\|^2 - |\alpha|^2 \end{aligned}$$

Hence, $|\alpha| = |\langle f, h \rangle| \leq \|f\|$. This completes the proof.

Theorem. (Minkowski inequality)

$\|f+g\| \leq \|f\| + \|g\|$

Proof:

$$\begin{aligned}
\|f+g\|^2 &= \langle f+g, f+g\rangle = \|f\|^2 + \langle f, g\rangle + \langle g, f\rangle + \|g\|^2 \\
&\leq \|f\|^2 + 2\|f\|\|g\| + \|g\|^2 \\
&= (\|f\| + \|g\|)^2
\end{aligned}$$