3.5 Pointwise, Uniform, Norm convergence, & Convergence in measure

Four important examples.

- Consider $\phi_n \in L^1(\mathbb{R}, d\mu)$ s.t. $\|\phi_n\| = \int_{\mathbb{R}} |\phi_n| d\mu = 1 \not\to 0$ but $\phi_n \to 0$ in some sense.
  - If $\heartsuit$ $\phi_n = \frac{1}{n}\chi_{(0,n)}$, then $\phi_n \to 0$ uniformly.
  - If $\diamondsuit$ $\phi_n = \chi_{(n,n+1)}$, then $\phi_n \to 0$ pointwise.
  - If $\spadesuit$ $\phi_n = n\chi_{(0,1/n)}$, then $\phi_n \to 0$ a.e.
- For each $k = 0, 1, 2, \cdots$, define a sequence $\clubsuit$ $\psi_{k,j} = \chi_{(j2^{-k},(j+1)2^{-k})}$ for $j = 0, \cdots, 2^k - 1$.
  1. Denote $\phi_1 = \psi_{0,0}$, $\phi_1 = \psi_{1,0}$, $\phi_2 = \psi_{1,1}$, $\phi_3 = \psi_{2,0}$, $\phi_4 = \psi_{2,1}$, $\phi_3 = \psi_{2,2}$, $\phi_3 = \psi_{2,3}$, $\phi_4 = \psi_{3,0}$.
  2. Then $\phi_1 = \chi_{(0,1)}$, $\phi_2 = \chi_{(0,2^{-1})}$, $\phi_3 = \chi_{(2^{-1},1)}, \cdots$
  3. $\|\psi_{k,j} - 0\| = 2^{-k} \to 0$, while $\phi_n(x) \not\to 0$ for any x.

Definition. (3.5.1: $f_n \to f$ (meas))

- $\{f_n\}$ *is said to* **converge in measure to** f *if*

$$\forall \epsilon > 0, \quad \lim_{n\to\infty} \mu(\{|f_n - f| \geq \epsilon\}) = 0$$

- $\{f_n\}$ *is said to be* **Cauchy sequence in measure** *if*

$$\forall \epsilon > 0, \quad \lim_{n,m\to\infty} \mu(\{|f_n - f_m| \geq \epsilon\}) = 0$$

- The sequences ♠ ♣, ♡ converge to 0 in measure. That is, $\phi_n = \frac{1}{n}\chi_{(0,n)}$, $\chi_{(0,1/n)}$, $\chi_{(j2^{-k},(j+1)2^{-k})} \to 0$ [meas].
- The sequence ♢ $\phi_n = \chi_{(n,n+1)}$ does not converge to 0 in measure.

Theorem. (3.5.2: f_n: Cauchy in meas $\Rightarrow$ $f_n \to^{\exists} f$ in meas)

Suppose $\{f_n\}$ is a Cauchy seq in measure. Then

- $\exists\ f \in \mathcal{M}_{able}$ *s.t.* $f_n \to f$ *in measure.*
- $\exists\ f_{n_k}$ *s.t.* $f_{n_k} \to f$ *a.e.*
- f *is uniquely determined a.e.*

Proof.

1. Choose a subsequence n_k such that $g_k = f_{n_k}$,

$$\mu(E_k) \leq 2^{-k}, \qquad E_k = \{|g_k - g_{k+1}| \geq 2^{-k}\}$$

2. Let $Z_k := \cup_{j=k+1}^{\infty} E_j$. Then $\mu(Z_k) \leq 2^{-k}$.
3. Let $Z = \cap_{k=1}^{\infty} Z_k$. Then $\mu(Z) = 0$.
4. Prove that $\lim_{k\to\infty} g_k(x) = {}^{\exists}f(x)$ for all $x \in X \setminus Z$
5. Prove that g_k converges to f uniformly on $X \setminus Z_N$ (N=1,2,...).

- Prove that $\lim_{k\to\infty} g_k(x) = {}^{\exists}f(x)$ for all $x \in X \setminus Z$

Proof.

1. For $x \in X \setminus Z_N$ and $j > i$, we have

$$\begin{aligned} |g_{N+i}(x) - g_{N+j}(x)| &\leq \sum_{k=i}^{j-1} |g_{N+k}(x) - g_{N+k+1}(x)| \\ &\leq \sum_{k=i}^{j-1} 2^{-N-k} \leq 2^{-N-i+1} \end{aligned}$$

2. $\therefore$ $g_k(x)$ is Cauchy sequence if $x \in X \setminus Z_N$, $N = 1, 2, \cdots$.
3. $\therefore$ $\lim_{k\to\infty} g_k(x)$ exists if $x \in X \setminus Z$.
4. Define $f : X \to \mathbb{R}$ by $f(x) = \lim_{k\to\infty} g_k(x)$ for $x \in X \setminus Z$ and $f(x) = 0$ for $x \in Z$.
5. Since $\mu(Z) = 0$, $g_k \to f$ *a.e.*.

- Prove that g_k converges to f uniformly on $X \setminus Z_N$ (N=1,2,...).

Proof. For $x \in X \setminus Z_N$, we have
$|g_{N+k}(x) - f(x)| = \lim_j |g_{N+k}(x) - g_{N+j}(x)| \leq 2^{-N-k+1}$,
that is,

$$\sup_{X \setminus Z_N} |g_{N+k} - f| \leq 2^{-N-k+1}$$

- Prove that f_n converges to f in measure.

Proof. For any $\epsilon > 0$, we have

$$\begin{aligned} \mu(\{|f - f_n| \geq \epsilon\}) &= \mu(\{|f - g_k| \geq \frac{\epsilon}{2}\}) + \mu(\{|g_k - f_n| \geq \frac{\epsilon}{2}\}) \\ &\to 0 \quad \text{as} \quad k, n \to \infty \end{aligned}$$

Corollary. (3.5.3)

If $f_n \to f$ in L^1, then there is a subsequence f_{n_k} such that $f_{n_k} \to f$ a.e.

Proof. For any $\epsilon > 0$,

$$\mu(\{|f_n - f| > \epsilon\}) \leq \frac{1}{\epsilon} \int |f_n - f| \, d\mu \to 0.$$

Hence, $f_n \to f$ in measure, and from the proof of Theorem 3.5.2 we can choose a subsequence n_k such that $g_k = f_{n_k}$,

$$\mu(E_k) \leq 2^{-k}, \qquad E_k = \{|g_k - g_{k+1}| \geq 2^{-k}\}.$$

From Theorem 3.5.2, $f_{n_k} = g_k \to f$ a.e..

Theorem. (3.5.4: Egorov's Theorem)

Let $\mu(E) < \infty$ and let $f_n \to f$ a.e.. Then, for any $\epsilon > 0$, there exist $F \subset E$ so that $\mu(E \setminus F) < \epsilon$ and $f_n \to f$ uniformly on F.

Proof. Since $f_n \to f$ *a.e.*, there exist $Z \subset E$ so that $\mu(Z) = 0$ and $f_n(x) \to f(x)$ for $x \in E \setminus Z$. It suffices to prove the theorem for the case when $Z = \emptyset$.

1. Let $E_{m,n} = \cap_{j=m}^{\infty} \{|f_j - f| < \frac{1}{n}\}$.
2. Then $\lim_{m\to\infty} \mu(E_{m,n}) = E$ for $n = 1, 2, \cdots$. (why?)
3. Hence, $\exists\ m_n$ s.t. $\mu(E \setminus E_{m_n,n}) \leq \epsilon 2^{-n}$.
4. Let $F = \cap_{n=1}^{\infty} E_{m_n,n}$. Then $\mu(E \setminus F) < \epsilon$. (why?)
5. Moreover, if $j > m_n$, then $\sup_{x\in F} |f_j(x) - f(x)| < \frac{1}{n}$. Hence, $f_n \to f$ uniformly on F.