Caratheodory Theorem

Definition. (2.2.1; Outer measure)

- *Let $(X, \mathcal{M}, \mu)$ be a measure space.*
    - *Recall*
        (i) *X is a set.*
        (ii) *$\mathcal{M}$ is a $\sigma-$algebra, that is, closed under a countable union and complementations.*
        (iii) *μ is a measure on $\mathcal{M}$, non-negative & countably additive .*
    - *A* **null set** *is a set N s.t. $\mu(N) = 0$*
    - *If $\sigma-$algebra $\mathcal{M}$ includes all null set, then μ is said to be* **complete.**
- *An outer measure on a non-empty set X is a set function μ^* defined on $\mathcal{P}(X)$ which is* **non-negative, monotone and countably subadditive.**

Why introduce the outer measure? Want to **describe a general constructive procedure for obtaining complete measure.**

Example of outer measure in $X = \mathbb{R}^2$

- $X = \mathbb{R}^2$, $\mathcal{E}$=the σ-algebra generated by the set of all open rectangles in $\mathbb{R}^2$, and define

$$\rho(E) = \text{ the area of } E, \qquad E \in \mathcal{E}$$

- $(X, \mathcal{E}, \rho)$ is a measure space but it may not be complete.
- This ρ is called **pre-measure**.
- For $A \subset X$, we define

$$\mu^*(A) = \inf\{\rho(E) \ : \ A \subset E, \ E \in \mathcal{E}\}.$$

Then μ^* is an outer measure.

Proposition. (2.2.2: Construction of outer measure μ^* on $\mathcal{P}(X)$)

Let $\mathcal{E} \subset \mathcal{P}(X)$ be an algebra of sets and $\rho : \mathcal{E} \to \mathbb{R}^+ \cup \{0\}$ an set valued function such that $\rho(\emptyset) = 0$. For $A \subset X$, we define

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} \rho(E_j) \ : \ A \subset \cup_{j=1}^{\infty} E_j, \ E_j \in \mathcal{E} \right\}.$$

Then μ^* is an outer measure.

Proof.

1. **Non-negative.** By its definition, $\mu^*(\emptyset) = 0$ and $\mu(A) \geq 0$ for $A \subset X$.
2. **Monotone.** If $A \subset B$ and $B \subset \cup_{j=1}^{\infty} E_j$, then $A \subset \cup_{j=1}^{\infty} E_j$ and $\mu^*(A) \leq \mu^*(B)$.
3. See the next page.

Proposition. (Continue...)

$\mu^*(A) = \inf\left\{\sum_{j=1}^{\infty}\rho(E_j)\ :\ A \subset \cup_{j=1}^{\infty}E_j,\ E_j \in \mathcal{E}\right\}$: outer measure.

Continue...

3. It remains to prove **countable subadditivity.**
   Let $A = \cup_{j=1}^{\infty} A_j$. Let $\epsilon > 0$ be given.
   - For each $j = 1, 2, \cdots,$ $\exists E_{jk} \in \mathcal{E}$ s .t.

$$A_j \subset \cup_{k=1}^{\infty} E_{jk} \ \ \& \ \ \sum_{k=1}^{\infty} \mu^*(E_{jk}) \leq \mu^*(A_j) + \epsilon 2^{-j}$$

and therefore

$$\mu^*(A) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mu^*(E_{jk}) \leq \sum_{j=1}^{\infty}\left(\mu^*(A_j) + \epsilon 2^{-j}\right) \leq \sum_{j=1}^{\infty}\mu^*(A_j) + \epsilon$$

Since ϵ is arbitrary small, $\mu^*(A) \leq \sum_{j=1}^{\infty}\mu^*(A_j)$.

Definition. (2.2.3: μ^*-measurable by Caratheodory)

Let μ^* be an outer measure on a set X. A subset $A \subset X$ is said to be **μ^*-measurable** *if*

$$\forall E \subset X, \quad \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

- This is SUPER CLEVER defintion!
- This definition provides a method of constructing a complete measure space $(X, \mathcal{M}, \mu^*)$ where $\mathcal{M}$ is the collection of all measurable sets.
- **Example.** The Lebesgue measure on $X = \mathbb{R}$ is an extension of the pre-measure defined by $\rho((a, b]) = b - a$.
  1. Let $X = \mathbb{R}$. Let $\mathcal{E}$ be the smallest $\sigma-$algebra generated by half-open intervals $(a, b]$. Then $(\mathbb{R}, \mathcal{E}, \rho)$ is an measure space.
  2. Define the outer measure μ^* as in Prop 2.2.2.
  3. Denote by $\mathcal{M}$ the collection of all measurable sets.
  4. Then $(\mathbb{R}, \mathcal{M}, \mu^*)$ is a complete measure space.

Theorem. (2.2.4: Caratheodory extension theorem)

Let μ^* is an outer measure on X. Let $\mathcal{M}$ be the collection of all measurable sets. Then $\mathcal{M}$ is $\sigma-$algebra and **the restriction of μ^* to $\mathcal{M}$ is a complete measure.**

Proof.

- Prove that $\mathcal{M}$ is $\sigma-$algebra. Easy.
- Prove that $(X, \mathcal{M}, \mu^*)$ is a measure space. Easy.
- Prove that μ^* is complete measure.
  Proof. If $\mu^*(A) = 0$, then for any $E \subset X$

  $$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \setminus A) \leq \mu^*(A) + \mu^*(E) = \mu^*(E)$$

  Hence, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$ for any $E \subset X$.
  Hence, $A \in \mathcal{M}$.

Lebesgue-Stieltjes measure

- **Example: Lebesgue-Stieltjes measure on $X = \mathbb{R}$.**
  - Let $\mathcal{E}$ be the algebra containing half open intervals $(a, b]$.
  - Define $\rho_F((a, b]) = F(b) - F(a)$ where $F : \mathbb{R} \to \mathbb{R}$ is a monotone increasing function.
  - ρ_F is a pre-measure measure on $\mathcal{E}$ but ρ_F is not complete.
  - Let μ^* be the outer measure defined as before.
  - Denote by $\mathcal{M}$ the collection of all measurable sets.
  - Denote by $\mu = \mu^*|_{\mathcal{M}}$ the restriction of μ^* on $\mathcal{M}$.
  - This μ is called a Lebesgue-Stieltjes measure generated by F.
- **Example: Lebesgue-Stieltjes measure on $X = \mathbb{R}^n$ or metric space.** The corresponding outer measure of Lebesgue measure μ is

$$\mu^*(A) = \inf \{\rho(U) \ : \ A \subset U, \ U \text{ open } \}$$

where ρ is a pre-measure defined on open sets in X. For example in $X = \mathbb{R}^2$, $\rho(U) =$ the volume of U.

Definition. (Metric Space (X, d) equipped with d =distance)

A metric space (M, d) *is a set* M *and a function* $d : M \times M \to R$ *such that*

1. $d(x, y) \geq 0$ *for all* $x, y \in X$.
2. $d(x, y) = 0$ *iff* $x = y$.
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$.
4. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y \in X$.

Example [Fingerprint Recognition] Let X be a data set of fingerprints in Seoul city police department.

- Motivation: Design an efficient access system to find a target.
- We need to define a **dissimilarity** function stating the distance between the data. The distance $d(x, y)$ between two data x and y must satisfy the above four rules.
- **Similarity queries.** For a given target $x^* \in X$ and $\epsilon > 0$, arrest all having finger print $y \in X$ such that $d(y, x^*) < \epsilon$.