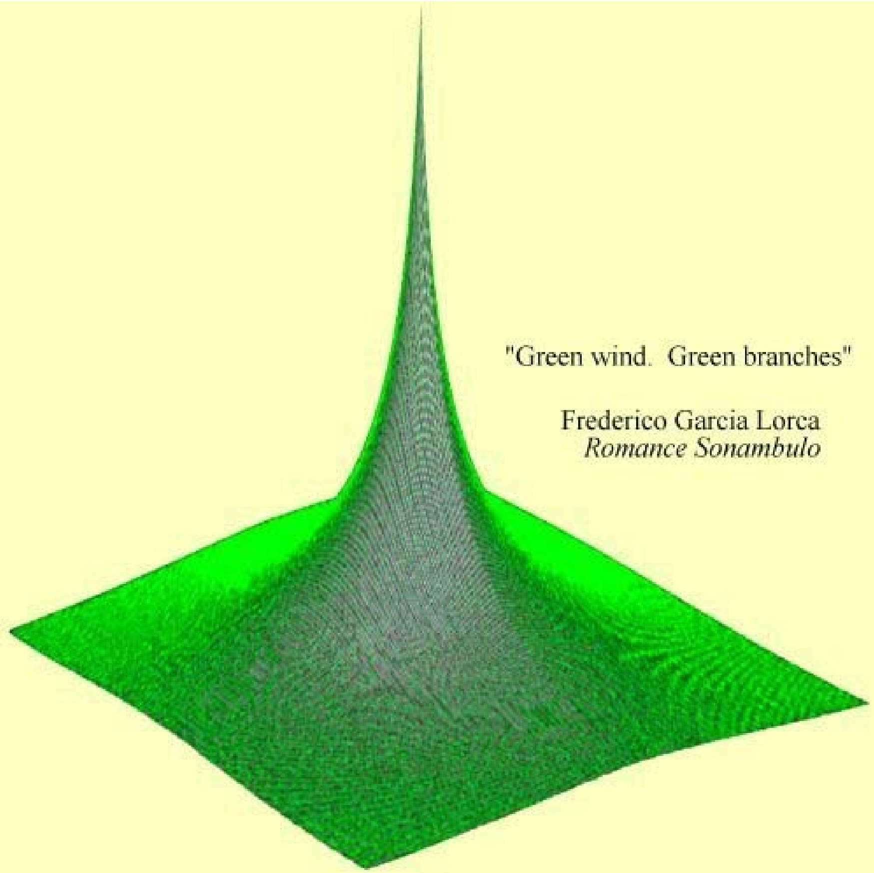


Green Function for Laplacian

Definition

Let Ω be a smooth domain in $\mathbb{R}^3$. The Green's function of the Laplace equation for the domain Ω is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} - H(\mathbf{r},\mathbf{r}'), \qquad \mathbf{r},\mathbf{r}' \in \Omega$$

where $H(\mathbf{r},\mathbf{r}')$ satisfies the following: for each $\mathbf{r}' \in \Omega$,

$$\begin{cases} -\nabla^2 H(\mathbf{r},\mathbf{r}') = 0 & \text{for } \mathbf{r} \in \Omega \\ H(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} & \text{for } \mathbf{r} \in \partial\Omega. \end{cases}$$

$$-\nabla_{\mathbf{r}}^2 H(r,r') = 0$$

$$\forall r, r' \in \Omega$$

$$H(r,r') = \frac{1}{4\pi|r-r'|}$$

- For a fixed $\mathbf{r}' \in \Omega$, the Green's function satisfies the following in the sense of distribution:

$$\begin{cases} -\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') & \text{for } \mathbf{r} \in \Omega \\ G(\mathbf{r},\mathbf{r}') = 0 & \text{for } \mathbf{r} \in \partial\Omega. \end{cases}$$

- When $\Omega = \mathbb{R}^3$, $H = 0$ and the Green's function $G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$.

$$-\nabla^2 G(r-r') = -\nabla^2 \frac{1}{4\pi|r-r'|} + \nabla^2 H(r,r') = \delta(r-r')$$

Theorem

We can express the solution of the Poisson's equation

$$\begin{cases} -\nabla^2 u = \rho & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

as

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' \qquad (\forall \mathbf{x} \in \Omega).$$

$$-\nabla_{\mathbf{r}}^2 G(r, r') = \delta(r - r')$$

$$\forall r \in \Omega$$

- The proof follows from the fact that

$$\begin{cases} -\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') & \text{for } \mathbf{r} \in \Omega \\ G(\mathbf{r}, \mathbf{r}') = 0 & \text{for } \mathbf{r} \in \partial\Omega. \end{cases}$$

- However, finding Green's function can be very very difficult if the domain is not a ball, half space, and domains with very special geometry. Solving the PDE using FEM is much much easier.

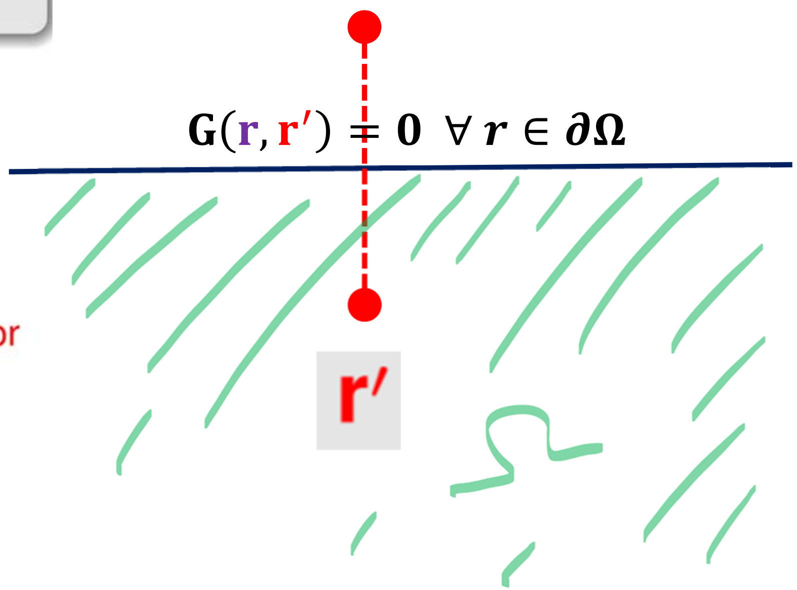
$$-\nabla_{\mathbf{r}}^2 u = -\int_\Omega \nabla_{\mathbf{r}}^2 G(r, r')\,\rho(r')\,dr' = \int_\Omega \delta(r - r')\ \rho(r')dr' = \rho(r)$$

Theorem (Green's function in half-space in 3D)

Let $\Omega := \{\mathbf{r} = (x, y, z) \ : \ z > 0\}$ be the upper half-space. Then, the Green's function of the half-space is given by

$$G(\mathbf{r};\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} - \underbrace{\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'_*|}}_{H(\mathbf{r},\mathbf{r}')}, \qquad \mathbf{r}'_* = (x', y', -z').$$

$\mathbf{r}'_* = (x', y', -z')$



- To find a Green's function, we need to find the function $H(\mathbf{r}, \mathbf{r}')$ such that, for each $\mathbf{r}' \in \Omega$,

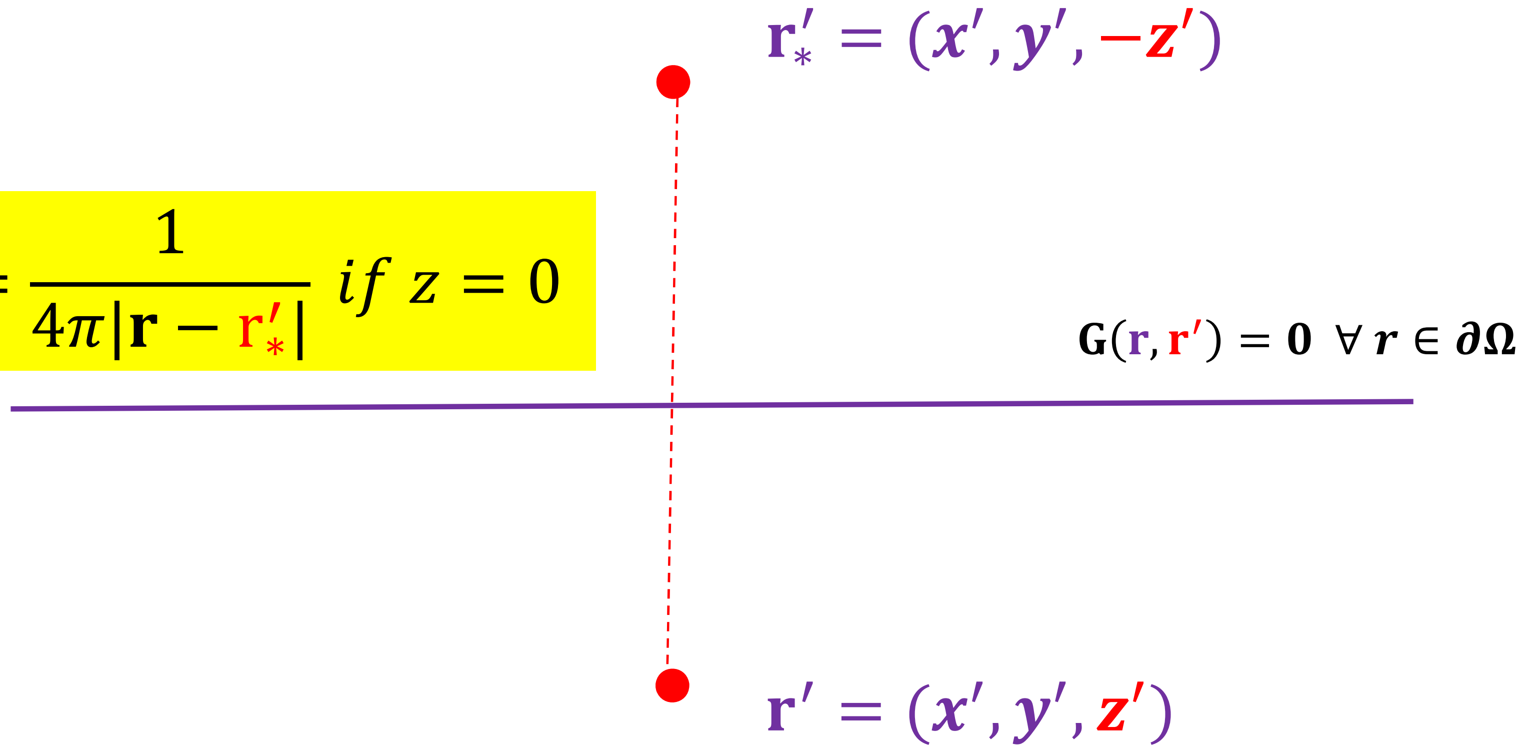
$$\begin{cases} -\nabla^2 H(\mathbf{r},\mathbf{r}') = 0 & \text{for } \mathbf{r} \in \Omega \\ H(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} & \text{for } \mathbf{r} \in \partial\Omega. \end{cases}$$

- The idea is to match the boundary value of the Green's function using the mirror image of a source in Ω. When the mirror is located at $\partial\Omega$, the point $\mathbf{r}'_* = (x', y, -z)$ is the mirror image point of $\mathbf{r}$ with respect to the surface $\partial\Omega$.
- With the aid of the mirror image point $\mathbf{r}'_* = (x', y', -z')$ of $\mathbf{r}'$, the function

$$\underbrace{\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}}_{\text{fundamental sol'n}} = \underbrace{\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'_*|}}_{H(\mathbf{r},\mathbf{r}'_*)} \qquad (\forall \mathbf{r}' \in \Omega, \ \ \mathbf{r} \in \partial\Omega)$$

Since $\mathbf{r}_*' \notin \Omega$, we have $-\nabla^2 \dfrac{1}{4\pi|\mathbf{r}-\mathbf{r}'_*|} = 0, \ \forall \mathbf{r} \in \Omega$

$$\frac{1}{4\pi|\mathbf{r}-r'|} = \frac{1}{4\pi|\mathbf{r}-\mathrm{r}'_*|} \ \ if\ z = 0$$



Theorem (Green's function in a ball)

The Green's function of the ball $\Omega = \{\mathbf{r} = (x, y, z) \;:\; |\mathbf{r}| < s\}$ *is given by*

$$G(\mathbf{r};\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} - \underbrace{\frac{s}{4\pi|\mathbf{r}'|\left|\mathbf{r} - \frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}\right|}}_{H(\mathbf{r},\mathbf{r}')}.$$

$$\boxed{\mathbf{r}'_* = \frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}.}$$

- The image point $\mathbf{r}'_*$ with respect to the sphere $\partial\Omega$ is given by

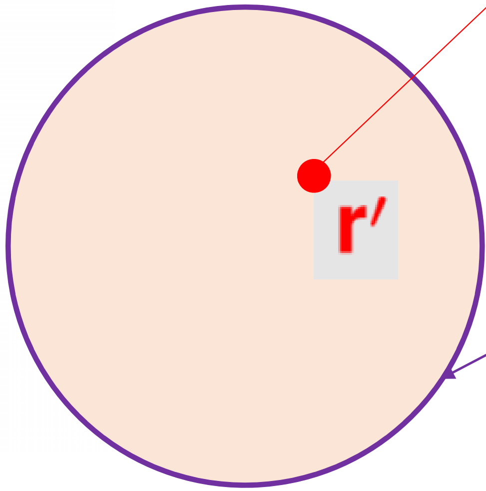
$$\mathbf{r}'_* = \frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}.$$

As a result,

$$H(\mathbf{r},\mathbf{r}') = \frac{s}{4\pi|\mathbf{r}'|\left|\mathbf{r} - \frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}\right|}$$

is harmonic w.r.t. $\mathbf{r}$ and

$$\frac{s}{4\pi|\mathbf{r}'|\left|\mathbf{r} - \frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}\right|} = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad \text{for all } |\mathbf{r}| = s.$$



$$\mathbf{G}(\mathbf{r},\mathbf{r}') = \mathbf{0}$$

$$\forall\, \boldsymbol{r} \in \boldsymbol{\partial\Omega}$$

Since $\mathbf{r}_*' \notin \Omega$, we have $-\nabla_{\mathbf{r}}^2 \dfrac{1}{4\pi|\mathbf{r}-\mathbf{r}'_*|} = 0, \;\forall \mathbf{r} \in \Omega$

Lemma

If G is a Green's function for the domain Ω, then it is symmetric:

$$G(\mathbf{x},\mathbf{y}) = G(\mathbf{y},\mathbf{x}) \qquad (\forall \mathbf{x},\mathbf{y} \in \Omega)$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$.

- Proof. Integrating by parts gives

$$\begin{aligned}
G(\mathbf{x},\mathbf{y}) &= \int_\Omega \delta(\mathbf{z}-\mathbf{x}) G(\mathbf{z},\mathbf{y}) d\mathbf{z} = -\int_\Omega \nabla^2_{\mathbf{z}} G(\mathbf{z},\mathbf{x}) \; G(\mathbf{z},\mathbf{y}) d\mathbf{z} \\
&= \int_\Omega \nabla_{\mathbf{z}} G(\mathbf{z},\mathbf{x}) \cdot \nabla_{\mathbf{z}} G(\mathbf{z},\mathbf{y}) d\mathbf{z} \\
&= -\int_\Omega G(\mathbf{z},\mathbf{x}) \cdot \nabla^2_{\mathbf{z}} G(\mathbf{z},\mathbf{y}) d\mathbf{z} = \int_\Omega G(\mathbf{z},\mathbf{x}) \; \delta(\mathbf{z}-\mathbf{y}) d\mathbf{z} \\
&= G(\mathbf{y},\mathbf{x}).
\end{aligned}$$

Theorem (Poisson's equation with Dirichlet BC)

For $f \in C^1(\partial\Omega)$, the solution of the Poisson's equation

$$\begin{cases} -\nabla^2 u = \rho & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$$

is expressed as

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' + \int_{\partial\Omega} K(\mathbf{r};\mathbf{r}')\, f(\mathbf{r}')\, dS_{\mathbf{r}'}, \quad \mathbf{r} \in \Omega$$

where the Poisson kernel $K(\mathbf{r},\mathbf{r}')$ is

$$K(\mathbf{r},\mathbf{r}') = -\mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}'), \qquad (\forall \mathbf{r}' \in \partial\Omega \;\; \& \;\; \forall \mathbf{r} \in \Omega).$$

- For a fixed $\mathbf{r} \in \Omega$, integration by parts gives

$$\begin{aligned} u(\mathbf{r}) &= \int_\Omega \delta(\mathbf{r}-\mathbf{r}')\, u(\mathbf{r}')d\mathbf{r}' = -\int_\Omega \nabla^2_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}')\, u(\mathbf{r}')d\mathbf{r}' \qquad (\text{ distributional sense}) \\ &= \int_{\partial\Omega} K(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\, dS_{\mathbf{r}'} + \int_\Omega \nabla_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}') \cdot \nabla u(\mathbf{r}')d\mathbf{r}' \\ &= \int_{\partial\Omega} K(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\, dS_{\mathbf{r}'} + \int_\Omega G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'. \end{aligned}$$

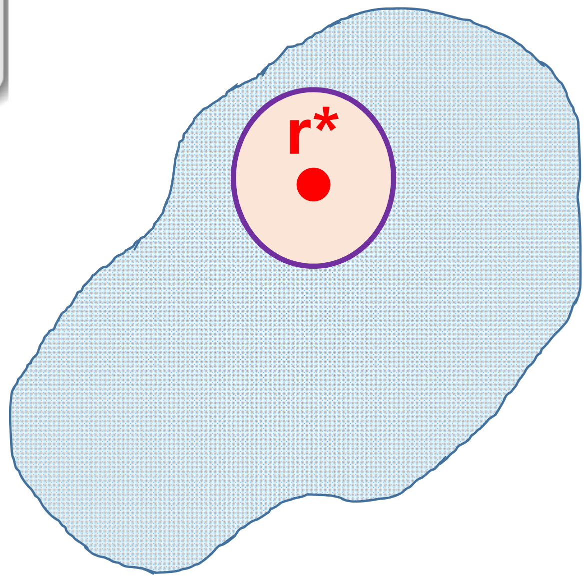
Theorem (Mean value property and maximum principle)

Assume that $u \in C^2(\bar{\Omega})$ *satisfies* $\nabla^2 u(\mathbf{r}) = 0$ *in* Ω*. For each* $\mathbf{r}^* \in \Omega$*,* $u(\mathbf{r}^*)$ *is the average of* u *over its neighboring sphere centered at* $\mathbf{r}^*$*:*

$$u(\mathbf{r}^*) = \frac{1}{4\pi s^2} \int_{\partial B_s(\mathbf{r}^*)} u(\mathbf{r}) dS_{\mathbf{r}} \qquad (\forall B_s(\mathbf{r}^*) \subset \Omega).$$

Moreover, u *can not have a strict local maximum or minimum at any interior point of* Ω*. In other words,*

$$\sup_{\partial\Omega} u = \sup_{\Omega} u \qquad \text{and} \qquad \min_{\partial\Omega} u = \min_{\Omega} u.$$



- Assume $B_s(\mathbf{r}^*) \subset \Omega$. If $K(\mathbf{r}, \mathbf{r}')$ is the Poisson Kernel for the ball $B_s(\mathbf{r}^*)$, a straightforward computation gives

$$K(\mathbf{r}, \mathbf{r}^*)|_{\mathbf{r} \in B_s(\mathbf{r}^*)} = \text{ constant} = \frac{1}{4\pi s^2}.$$

Then, the mean value property follows from the representation formula

$$u(\mathbf{r}) = \int_{\Omega} G(\mathbf{r}; \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' + \int_{\partial\Omega} K(\mathbf{r}; \mathbf{r}')\, f(\mathbf{r}')\, dS_{\mathbf{r}'}$$

with $\rho = 0$ and the above identity. Moreover, the mean value property gives the maximum principle.

Theorem

Denote $B_s = \{|\mathbf{r}| < s\}$*. If* $u \in C^2(\bar{B}_s)$ *satisfies* $-\nabla^2 u = \rho$ *in* B_s*, then*

$$u(0) = ave_{\partial B_s} u \;+\; \int_{B_s} \frac{1}{4\pi}\left(\frac{1}{|\mathbf{r}|} - \frac{1}{s}\right)\rho(\mathbf{r})d\mathbf{r}$$

where $ave_{\partial B_s} u = \frac{1}{4\pi s^2}\int_{\partial B_s} u(\mathbf{r})dS_{\mathbf{r}}$*. In particular,*

$$\nabla^2 u = \rho \geq 0 \;\; \text{in } B_s \Rightarrow \;\; u(0) \leq ave_{\partial B_s} u,$$
$$\nabla^2 u = \rho \leq 0 \;\; \text{in } B_s \Rightarrow \;\; u(0) \geq ave_{\partial B_s} u.$$

- The identity follows from the representation formula

$$u(\mathbf{r}) = \int_{\Omega} G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' \;+\; \int_{\partial\Omega} K(\mathbf{r};\mathbf{r}')\, f(\mathbf{r}')\, dS_{\mathbf{r}'}$$

Theorem

Let $\rho \in C(\bar{\Omega})$ and $f \in C^1(\partial\Omega)$. Suppose that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is the solution of

$$\begin{cases} -\nabla^2 u = \rho & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$$

Then, u is characterized as the unique solution that minimizes the Dirichlet functional

$$\Phi(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 - \rho v d\mathbf{x}$$

within the set $\mathcal{A} := \{v \in C^2(\Omega) \cap C^1(\bar{\Omega}) \ : \ v|_{\partial\Omega} = f\}$.

- Suppose $w = \arg\min_{v \in \mathcal{A}} \Phi(v)$. Since $\Phi(w) \le \Phi(w + t\phi)$ for any $\phi \in C_0^1(\Omega)$ and any $t \in \mathbb{R}$,

$$0 = \frac{d}{dt}\Phi(w + t\phi)\Big|_{t=0} = \int_\Omega \nabla w \cdot \nabla \phi - \rho\phi d\mathbf{x} \qquad \forall \phi \in C_0^1(\Omega).$$

Integration by parts yields

$$\int_\Omega (\nabla^2 w + \rho)\phi d\mathbf{x} = 0 \qquad \forall \phi \in C_0^1(\Omega),$$

which leads to $\nabla^2 w + \rho = 0$ in Ω.

Energy Minimization Approach

If u is the solution of $-\nabla u = \rho$ in Ω with $u|_{\partial\Omega} = f$, then u minimizes the energy functional $\Phi(u)$ within the class $\mathcal{A} := \{w \in C^2(\bar{\Omega}) : u|_{\partial\Omega} = f\}$ where the energy functional is defined by

$$\Phi(u) := \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - \rho u \right) d\mathbf{r}.$$

Proof.

- For all $\phi \in \mathcal{A}$,

$$0 = \int_\Omega (-\nabla^2 u - \rho)(u - \phi) dx = \int_\Omega \nabla u \cdot \nabla (u - \phi) - \rho (u - \phi).$$

- For all $\phi \in \mathcal{A}$,

$$\int_\Omega |\nabla u|^2 - \rho u d\mathbf{r} = \int_\Omega \nabla u \cdot \nabla \phi - \rho \phi d\mathbf{r}.$$

- Since $|\nabla u \cdot \nabla \phi| \leq \frac{1}{2} |\nabla u|^2 + \frac{1}{2} |\nabla \phi|^2$,

$$\Phi(u) \leq \Phi(\phi) \qquad \text{for all } \phi \in \mathcal{A}.$$