# Minimizing Loss Function

For a given $\boldsymbol{\rho} =$  , what is the solution $\boldsymbol{u}_*$ satisfying

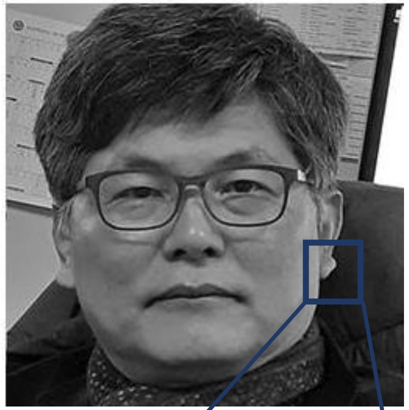
$$\mathbf{u}_* = \underset{\boldsymbol{u} \in \boldsymbol{some\ set}}{\mathbf{argmin}}\ \boldsymbol{\Phi}(\boldsymbol{u}) \quad \text{where } \Phi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \rho u$$

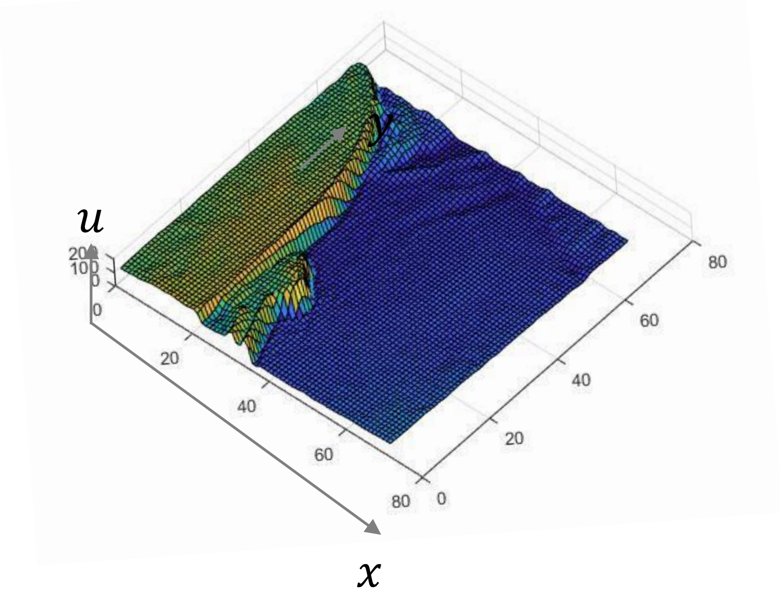
Answer: $\mathbf{u}_* =$ 

# Understanding PDEs using 2D images as examples

- A function $u(x, y)$ of two variables $(x, y)$ can be represented as a grayscale image.
- The grayscale image is represented as a matrix $(u_{ij})$, with each element corresponding to one image pixel.

$$\begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{m1} & \cdots & u_{mn} \end{bmatrix} =$$






$$u(x, y) \approx u_{ij}$$

white-255
black-0


| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 108 | 148 | 168 | 105 | 105 | 170 | 133 | 25 | 23 | 41 |
| 114 | 155 | 152 | 92 | 127 | 169 | 82 | 12 | 31 | 36 |
| 122 | 161 | 135 | 87 | 147 | 145 | 34 | 21 | 46 | 37 |
| 149 | 148 | 112 | 112 | 157 | 104 | 17 | 41 | 35 | 38 |
| 154 | 147 | 101 | 128 | 139 | 64 | 19 | 37 | 38 | 35 |
| 152 | 131 | 99 | 146 | 106 | 21 | 28 | 35 | 42 | 35 |
| 142 | 110 | 117 | 147 | 63 | 6 | 39 | 37 | 44 | 39 |
| 126 | 105 | 142 | 122 | 24 | 18 | 45 | 39 | 41 | 41 |
| 111 | 113 | 155 | 84 | 5 | 32 | 42 | 37 | 37 | 39 |
| 110 | 130 | 152 | 50 | 12 | 36 | 36 | 36 | 39 | 38 |

Each pixel is assigned a value of grayscale level between 0 and 255

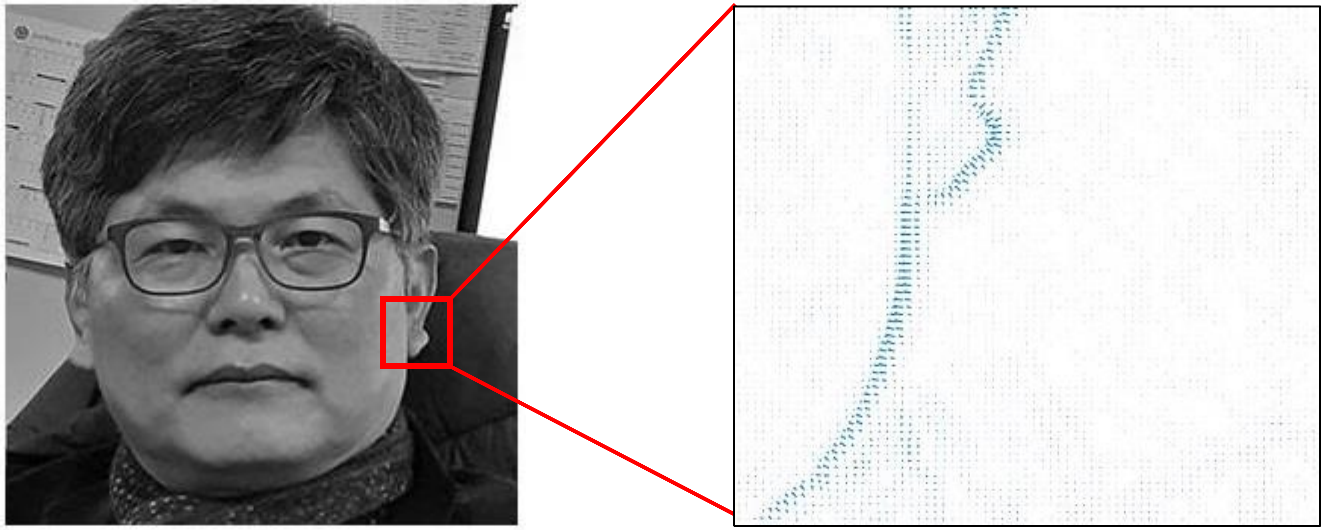
# Directional Derivative

The right figure shows the image of the directional derivative $(2,3)\cdot\nabla\mathbf{u} = 2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y}$.

$$\frac{\partial u}{\partial x} = u_{i+1,j} - u_{i,j} \quad \& \quad \frac{\partial u}{\partial y} = u_{i,j+1} - u_{i,j}$$

$$\boldsymbol{u} = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{m1} & \cdots & u_{mn} \end{bmatrix}$$

$$\nabla \boldsymbol{u} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$



$$(2,3)\cdot\nabla\mathbf{u}$$

If $\mathbf{u}_* = \underset{\boldsymbol{u}}{\mathbf{argmin}}\, \boldsymbol{\Phi}(\boldsymbol{u})$ , $\mathbf{u}_*$ is a critical point of the function $\boldsymbol{\Phi}(\boldsymbol{u})$, that is,

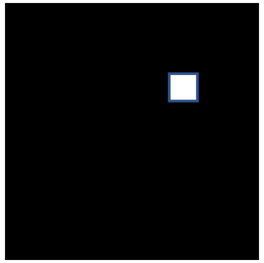
$$\mathbf{0} = \frac{\mathbf{d}}{\mathbf{dt}} \boldsymbol{\Phi}(\boldsymbol{u}_* + \boldsymbol{t\phi}) \Big|_{t=0} \qquad \textit{for all } \phi$$

$$= \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \frac{1}{2} |\nabla(u_* + t\phi)|^2 - \rho(u_* + t\phi) dr \Big|_{t=0}$$

$$= \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \frac{1}{2} |\nabla u_*|^2 + \mathrm{t}\nabla u_* \cdot \nabla\phi + t^2|\nabla\phi|^2 - \rho u_* - t\rho\phi \,\mathrm{dr} \Big|_{\mathrm{t}=0}$$

$$= \int_\Omega \nabla u_* \cdot \nabla\phi - \rho\phi = \int_\Omega \boxed{(-\nabla^2 u_* - \rho)}\, \phi$$

The loss minimization problem is changed to a linear algebra $-\nabla^2 u = \rho$ .

$$\frac{\mathbf{d}}{\mathbf{dt}} \boldsymbol{\Phi}($$  $$+\boldsymbol{t}$$  $$)$$

# Poisson’s equation

This image $u(x,y) \approx u_{ij}$ can be viewed as a solution of Poisson’s equation

$$-\nabla^2 u = \rho \quad in \;\; \Omega$$

The sparse data $\rho$ contains almost the full information of the image $u$.

$u$

$\dfrac{\partial u}{\partial x}$

$\dfrac{\partial u}{\partial y}$

$\rho = \nabla^2 u$

## Theorem

*Let $\rho \in C(\bar{\Omega})$ and $f \in C^1(\partial\Omega)$. Suppose that $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is the solution of*

$$\begin{cases} -\nabla^2 u = \rho & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$$

*Then, $u$ is characterized as the unique solution that minimizes the Dirichlet functional*

$$\Phi(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 - \rho v d\mathbf{x}$$

*within the set $\mathcal{A} := \{v \in C^2(\Omega) \cap C^1(\bar{\Omega}) \ : \ v|_{\partial\Omega} = f\}$.*

- Suppose $w = \arg\min_{v\in\mathcal{A}} \Phi(v)$. Since $\Phi(w) \leq \Phi(w + t\phi)$ for any $\phi \in C_0^1(\Omega)$ and any $t \in \mathbb{R}$,

  $$0 = \frac{d}{dt}\Phi(w + t\phi)\bigg|_{t=0} = \int_\Omega \nabla w \cdot \nabla\phi - \rho\phi d\mathbf{x} \qquad \forall\phi \in C_0^1(\Omega).$$

  Integration by parts yields

  $$\int_\Omega (\nabla^2 w + \rho)\phi d\mathbf{x} = 0 \qquad \forall\phi \in C_0^1(\Omega),$$

  which leads to $\nabla^2 w + \rho = 0$ in $\Omega$.

# Energy Minimization Approach

If $u$ is the solution of $-\nabla u = \rho$ in $\Omega$ with $u|_{\partial\Omega} = f$, then $u$ minimizes the energy functional $\Phi(u)$ within the class $\mathcal{A} := \{w \in C^2(\bar{\Omega}) : u|_{\partial\Omega} = f\}$ where the energy functional is defined by

$$\Phi(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \rho u\right) d\mathbf{r}.$$

Proof.

- For all $\phi \in \mathcal{A}$,

$$0 = \int_\Omega (-\nabla^2 u - \rho)(u - \phi)dx = \int_\Omega \nabla u \cdot \nabla(u - \phi) - \rho(u - \phi).$$

- For all $\phi \in \mathcal{A}$,

$$\int_\Omega |\nabla u|^2 - \rho u d\mathbf{r} = \int_\Omega \nabla u \cdot \nabla \phi - \rho\phi d\mathbf{r}.$$

- Since $|\nabla u \cdot \nabla\phi| \leq \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla\phi|^2$,

$$\Phi(u) \leq \Phi(\phi) \qquad \text{for all } \phi \in \mathcal{A}.$$