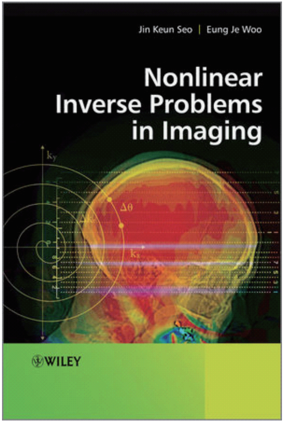
Partial Differential Equations



Part IV: Euler-Lagrange Equations

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Optimization in finite dimensional vector space

Engineering design is guided by a variety of optimization constraints, such as performance, safety, and cost. Minimization problems arise in the fitting data via least squares approximations.

Throughout this section, we want to minimize the objective function $J(\mathbf{u}) = J(u_1, u_2, \cdots, u_n)$ (the energy, cost, entropy, performance, etc.) within the open subset W of $\mathbb{R}^n$.

Minimization

Assume $\mathbf{u}_* \in W = \mathbb{R}^n$ is a global minimum of the objective function $J(\mathbf{u})$ on the domain W, that is,

$$J(\mathbf{u}_*) \leq J(\mathbf{v}) \qquad \text{for all} \quad \mathbf{v} \in W.$$

Then $\mathbf{u}_*$ is a critical point of J:

$$0 = \frac{d}{dt} J(\mathbf{u}_* + t\mathbf{v})|_{t=0} = \nabla J(\mathbf{u}_*) \cdot \mathbf{v} = \langle \nabla J(\mathbf{u}_*),\ \mathbf{v} \rangle \qquad \text{for all } \mathbf{v} \in \mathbb{R}^n.$$

The left side of the above identity is known as the directional derivative of J with respect to $\mathbf{v}$ that is defined as

$$\langle \nabla J(\mathbf{u}_*),\ \mathbf{v} \rangle = \sum_{j=1}^{n} \partial_j J(\mathbf{u}_*)\ v_j.$$

Euler-Lagrange equation: Minimization in the infinite dimensional context.

Now, let us learn how to compute the gradient of $J(u)$ on a subset W of the Hibert space $H^1(a,b) = \{u \mid \int_a^b |u|^2 + |u'|^2 dx < \infty\}$, an infinite dimensional function space.

Minimization problem in Hilbert space $H^1((a,b))$

Let $\sigma(x) \in C([a,b])$ and $a > 0$. For a given $f \in L^2(a,b)$ and $\alpha, \beta \in \mathbb{R}$, suppose u_* is a minimizer of the following minimization problem:

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_a^b |f(x) - u(x)|^2 + \sigma(x)|u'(x)|^2 dx \\ \text{subject to the constraint} & \\ & u \in W = \{u \in H^1(a,b) \mid u(a) = \alpha, u(b) = \beta\} \end{array} \right.$$

Then u_* satisfies

$$-(\sigma(x)u_*'(x))' + (u_*(x) - f(x)) = 0 \qquad a < \forall x < b$$

with the boundary condition $u_*(a) = \alpha, u_*(b) = \beta$.

Proof.

Continue...

- In this Hilbert space H^1, the inner product and the distance (or metric) between two functions u and v are given respectively by

$$\langle u,\ v\rangle = \int_a^b uv + u'v'\,dx \quad \& \quad \|u - v\| = \sqrt{\langle u - v,\ u - v\rangle}.$$

- The goal is to investigate a minimizer $u_* \in W$ satisfying

$$J(u_*) \le J(v) \qquad \text{for all } v \in W.$$

- This is equivalent to

$$J(u_*) \le J(u_* + tv) \qquad \forall t \in \mathbb{R},\ \ \forall v \in H_0^1(a, b)$$

where $H_0^1(a, b) := \{u \in H^1(a, b) : u(a) = 0 = u(b)\}$. (Why?)

- Hence,

$$\frac{d}{dt} J(u_* + tv)|_{t=0} = 0 \quad \forall t \in \mathbb{R}, \quad \forall v \in H_0^1.$$

Continue...

- By the chain rule and integrating by parts,

$$\begin{aligned}\frac{d}{dt}J(u+tv)|_{t=0} &= \lim_{t\to 0}\frac{1}{t}\left(J(u+tv)-J(u)\right)\\ &= \lim_{t\to 0}\frac{1}{t}\int_a^b \left[2tv(u-f)+2t\sigma u'v'+t^2(v^2+\sigma v'^2)\right]dx\\ &= \int_a^b [2v(u-f)+2\sigma u'v']\,dx\\ &= 2\int_a^b [(u-f)-(\sigma u')']\;v\,dx\end{aligned}$$

  This can be viewed as a directional derivative of J in the direction v -*the direction in which the derivative is computed*.

- Since this holds for all function $v \in H_0^1$, we have

$$\begin{aligned}&\frac{d}{dt}J(u_*+tv)|_{t=0}=0 && \forall v\in H_0^1\\ \Leftrightarrow\quad &\int_a^b [(u_*-f)-(\sigma u_*')']\;v\,dx && \forall v\in H_0^1\\ \Leftrightarrow\quad &-(\sigma u_*')'+(u_*-f)=0 && a<\forall x<b.\end{aligned}$$

  The last one is the Euler-Lagrange equation of J.

  ⋇ One can define the infinite dimensional gradient $\nabla J(u)$ as

$$\nabla J(u) \;=\; -(c(x)u')'(x)+u(x)-f(x).$$

Example: Curve minimization problem joining two point (a, α)

Consider the curve minimization problem joining two point (a, α) and (b, β):

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) = \int_a^b \sqrt{1 + (u')^2} dx \\ \text{within the set} & W = \{u \in W^{1,1}(a, b) \mid u(a) = \alpha, u(b) = \beta\} \end{array} \right.$$

Here, $W^{1,1}(a, b) = \{u \mid \int_a^b \sqrt{u^2 + (u')^2} dx < \infty\}$. (This is a kind of Sobolev space or a Banach space.) If u_* is a minimizer, then

$$u_*''(x) = 0 \qquad a < x < b.$$

- For all $v \in H_0^1$, we have

$$\begin{aligned} \frac{d}{dt} J(u + tv)|_{t=0} &= \frac{d}{dt} \int_a^b \sqrt{1 + (u' + tv')^2} dx \Big|_{t=0} \\ &= \int_a^b \frac{2u'v' + 2tv'^2}{\sqrt{1 + (u' + tv')^2}} dx \Big|_{t=0} = \int_a^b \frac{2u'v'}{\sqrt{1 + (u')^2}} dx \\ &= 2 \int_a^b \frac{d}{dx} \left[\frac{u'}{\sqrt{1 + (u')^2}} \right] v dx \\ &= 2 \int_a^b \frac{u''[\sqrt{1 + (u')^2} - 2u'^2]}{1 + (u')^2} \, v \, dx \end{aligned}$$

Continue...

- Since the above identity holds for all $v \in H_0^1$, the minimizer u_* satisfies Euler-Lagrange

$$\frac{u_*''[\sqrt{1+(u_*')^2} - 2u_*'^2]}{1+(u_*')^2} = 0 \qquad a < x < b$$

and hence

$$u_*''(x) = 0 \qquad a < x < b.$$

Hence, the straight line joining (a, α) and (b, β) is the minimizer.

Dirichelt Problem

Let Ω be a domain in $\mathbb{R}^2$. Consider the minimization problem

$$\left|\left| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega |\nabla u(x,y)|^2 dxdy \\ \text{within the set} & W := \{u \in H^1(\Omega) : u|_{\partial\Omega} = f(x,y)\} \end{array} \right.$$

Here, $H^1(\Omega) = \{u : \int_\Omega |u|^2 + |\nabla u|^2 dxdy < \infty\}$. Then the minimizer u satisfies the Dirichlet problem:

$$\begin{cases} \nabla^2 u(x,y) = 0 & (x,y) \in \Omega \\ u|_{\partial\Omega} = f & (\text{ the prescribed boundary potential}) \end{cases}$$

- For all $v \in H_0^1(\Omega) := \{v \in H^1(\Omega) \ : \ u|_{\partial\Omega} = 0\}$,

$$\begin{aligned} 0 &= \frac{d}{dt} J(u+tv)|_{t=0} = 2\int_\Omega \nabla u \cdot \nabla v \; dxdy \\ &= -2\int_\Omega \nabla^2 uv \; dxdy \end{aligned}$$

Since this hold for all $v \in H_0^1(\Omega)$, we have

$$0 = \nabla^2 u \qquad \text{in } \Omega$$

Minimal Surfaces

Consider the minimal surface problem

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega \sqrt{1+u_x^2+u_y^2}\,dxdy \\ \text{within the set} & W := \{u \in W^{1,1}(\Omega) : u|_{\partial\Omega} = f\} \end{array} \right.$$

Here, $W^{1,1}(\Omega) = \{u : \int_\Omega |u| + |\nabla u| dxdy < \infty\}$. If a minimizer u exist, u satisfies the following nonlinear problem with the Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = f & (\text{ Dirichlet boundary}) \end{cases}$$

- For all $v \in W_0^{1,1}(\Omega) := \{v \in W^{1,1}(\Omega) : v|_{\partial\Omega} = 0\}$,

$$\begin{aligned} 0 &= \frac{d}{dt} J(u+tv)|_{t=0} = \int_\Omega \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nabla v\; dxdy \\ &= \int_\Omega \left[-\nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) \right] v\; dxdy \end{aligned}$$

  Since this hold for all $v \in W_0^{1,1}(\Omega)$, we have

$$0 = \nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) \qquad \text{in } \Omega$$

Total Variation Minimization

Consider the minimization problem

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega |\nabla u(x,y)| + |u - u_0|^2 dxdy \\ \text{within the set} & W := \{u \in W^{1,1}(\Omega) : \mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0\} \end{array} \right.$$

If a minimizer u exist, u satisfies the following nonlinear problem with zero Neumann boundary condition:

$$\begin{cases} \nabla \cdot (\frac{\nabla u}{|\nabla u|}) = 2(u - u_0) & \text{in} \quad \Omega \\ \mathbf{n} \cdot \nabla u|_{\partial\Omega} = 0 & \text{(insulating boundary)} \end{cases}$$

- For all $v \in W^{1,1}(\Omega)$,

$$\begin{aligned} 0 &= \frac{d}{dt} J(u + tv)|_{t=0} = \int_\Omega \frac{\nabla u}{|\nabla u|} \cdot \nabla v + 2(u - u_0)v \, dxdy \\ &= \int_\Omega \left[-\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + 2(u - u_0) \right] v \, dxdy \end{aligned}$$

Since this hold for all $v \in W^{1,1}(\Omega)$, we have

$$0 = -\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + 2(u - u_0) \qquad \text{in } \Omega$$