Lebesgue-Radon-Nikodym Theorem

The goal is to understand the main structure of the following theorem which is a remarkable achievement.

Theorem. (5.3.6: Radon-Nikodym, Riesz representation)

Let μ and ν be $\sigma-$finite measures on a set X, and suppose that $\nu \ll \mu$ (ν is absolute continuous w.r.t.μ). Then

$$\exists \;\; f \in L^1(X, d\mu) \;\; s.t. \;\; \nu(E) = \int_E f \; d\mu$$

for all $E\mathcal{M}$ for which $\nu(E) < \infty$

- **Definition** : $\nu \ll \mu$ iff $\mu(E) = 0 \Rightarrow \nu(E) = 0$ for all $E \in \mathcal{M}$.
- **Definition** : μ and ν are mutually singular, writing $\mu \perp \nu$, iff

$$\exists \;\; E, F \in \mathcal{M} \;\; s.t. \;\; X = E \cup F, \; E \cap F = \emptyset, \;\; \mu(E) = 0 = \nu(F).$$

To prove Radon-Nikodym thm, we need to understand several concepts.

Throughout this section, X is a metric space.

Definition.

- *A real valued measure ν, defined on Borel subset of X, is called a signed measure if it can be written in the form $\nu = \mu_1 - \mu_2$ where μ_1 and μ_2 are positive Borel measures, at least one of which is finite.*
- *Define* **the total variation of** ν, *denoted by* $|\nu|$, *to be*

$$|\nu|(E) = \sup \left\{ \sum_j |\nu(E_j)| \right\}$$

where the sup is taken over all disjoint collection $\{E_j\}$ such that $E = \cup_j E_j$.

Theorem. (5.2.1: Jordan decomposition)

If ν is a signed measure, then its total variation $|\nu|$ is a positive, countably additive measure. Moreover, $\nu^+ = \frac{|\nu|+\nu}{2}$ and $\nu^- = \frac{|\nu|-\nu}{2}$ are positive measures. We thus have the decompositions:

$$\nu = \nu^+ - \nu^- \text{ (Jordan decomposition) } \quad \& \quad |\nu| = \nu^+ + \nu^-$$

Proof. It is easy to prove the followings:

1. $|\nu|(\emptyset) = 0$ and $|\nu|$ is monotone and finitely additive.
2. If $\{E_j\}$ is a countable partition of E, then $|\nu|(\cup E_j) = \sum |\nu|(E_j)$.
3. ν^+ and ν^- are positive measures.

Theorem. (5.2.3: Hahn Decomposition)

Let $\nu = \nu^+ - \nu^-$ be the Jordan decomposition of the signed measure ν. Then, $\nu^+ \perp \nu^-$, and the exist $P, N \in \mathcal{M}$ s.t.

$$P \cup N = X, \quad P \cap N = \emptyset, \quad \nu^+(N) = \nu^-(P) = 0$$

Moreover, $\nu(E) \geq 0, \ \forall \ E \subset P$, while $\nu(E) \leq 0, \ \forall \ E \subset N$.

Proof. Assume that $\nu^+(X) < \infty$ and let $\{E_j\}$ be a sequence s.t. $\nu(E_j) > \nu^+(X) - 2^{-j}$.

1. For $A \subset E_j^c$,

$$\nu(A) \leq \nu^+(E_j^c) = \nu^+(X) - \nu^+(E_j) \leq \nu^+(X) - \nu(E_j) \leq 2^{-j}.$$

2. $\nu^-(E_j) = -\nu(E_j) + \nu^+(E_j) \leq -\nu(E_j) + \nu^+(X) < 2^{-j}$
3. Let $\limsup_j E_j = P$ & $\liminf_j E_j^c = N$
4. Then $\nu^+(N) = 0 = \nu^-(P)$.

Theorem. (5.3.2)

Let ν be finite and μ a positive measure on $(X, \mathcal{M})$. Then

$$\nu \ll \mu \quad \textit{iff} \quad \forall\, \epsilon > 0 \quad \exists\, \delta > 0 \;\; \textit{s.t.} \;\; \mu(E) < \delta \;\Rightarrow\; |\nu(E)| < \epsilon$$

Proof. By Jordan decomposition, it suffices to prove the result when $\nu = \nu^+$. The sufficiency ($\Leftarrow$) is immediate; so we have only to establish its necessity.

1. To derive a contradiction, suppose $\nu \ll \mu$ but that $\epsilon - \delta$ condition fails.
2. Then $\exists\, \epsilon > 0$ and $\exists\, \{E_j\}$ s.t. $\nu(E_j) > \epsilon$ while $\mu(E_j) < 2^{-j}$.
3. Let $E = \limsup_j E_j = \cap_{k \geq 1} \cup_{j \geq k} E_j$.
4. By Fatou's lemma, $\nu(E) \geq \limsup_j \nu(E_j) \geq \epsilon$.
5. By monotone convergence theorem, $\mu(E) = \lim_k \mu(\cup_{j \geq k} E_j) \leq \lim_k \sum_{j=k}^{\infty} \mu(E_j) \leq \lim_k \sum_{j=k}^{\infty} 2^{-j} = \lim_k 2^{-k+1} = 0$.
6. From 4 and 5, ν is not absolutely continuous w.r.t. μ.

Theorem. (5.3.3)

Let $f \in L^1(X, d\mu)$, where μ is a positive measure. Define $\nu(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Then

$$\forall \, \epsilon > 0 \quad \exists \, \delta > 0 \;\; s.t. \;\; \mu(E) < \delta \; \Rightarrow \; \left| \int_E f \, d\mu \right| \; < \; \epsilon$$

Proof.

1. Define $\nu(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$.
2. Then $\nu \ll \mu$.
3. The result follows from 2 and 5.3.2

Lemma. (5.3.5)

Let ν be finite and positive, and put

$$\mathcal{F} = \left\{ f \in L^1(d\mu) \ : \ \nu(E) \geq \int_E f \ d\mu \ \ \textit{for all} \ \ E \in \mathcal{M} \right\}$$

Then $\exists$ unique $g \in \mathcal{F}$ s.t.

$$\int g d\mu \ = \ \sup_{f \in \mathcal{F}} \int \ f \ d\mu$$

Proof. The sup makes sense since $\mathcal{F} \neq \emptyset$ by Lemma 5.3.4.

1. Let $M = \sup_{f \in \mathcal{F}} \int \ f \ d\mu$ and let $\{f_j\}$ be a sequence in $\mathcal{F}$ s.t. $\int f_j d\mu \to M$.
2. Let $g_n = \max\{f_1, \ f_2, \cdots, f_n\}$. Then $g_n \nearrow$ and $M = \int \lim_j g_j \ d\mu$ by monotone convergence thm.

Continue the proof

4. To prove $g_n \in \mathcal{F}$, $\int_E \; g_n \; d\mu \le \nu(E)$.
   Proof.
   - Let $E_1 = E \cap \{g_n = f_1\}$ and
     $E_j = E \cap \{g_n = f_j\} \setminus \cup_{k=1}^{j-1} E_k,\; j = 2, \cdots, n$.
   - Then $E = \cup_{j=1}^{n} E_j$ and E_j are disjoint. Hence,

$$\int_E \, g_n \; d\mu \;=\; \sum_{j=1}^{n} \int_{E_j} f_j \; d\mu \le \; \sum_{j=1}^{n} \nu(E_j) \;=\; \nu(E)$$

4. From 4 and monotone convergence theorem, $g = \lim_j g_j \in \mathcal{F}$.

Theorem. (5.3.6: Radon-Nikodym)

Let μ and σ be $\sigma-$finite measures on a set X, and suppose that $\nu \ll \mu$. Then $\exists\ f \in L^1(X, d\mu)$ s.t.

$$\nu(E) = \int_E f \ d\mu \quad \textit{for all } \nu(E) < \infty \ \ (E \in \mathcal{M})$$

Proof.

1. From Jordan decomposition, the proof can be reduced to the case in which μ, ν are positive and finite.
2. Let f be the maximal function generated by Lemma 5.3.5.
3. Then $\nu_1(E) = \nu(E) - \int_E f d\mu$ satisfies $\nu_1 \geq 0$ and $\nu_1 \ll \mu$.
4. If $\nu_1 \neq 0$, $\exists\ \epsilon > 0$ & E' s.t. $\nu_1(E') > 0$ and

$$\nu_1(E'') \geq \epsilon\ \mu(E'') \quad \text{for } E'' \subset E' \qquad \text{(Why?)}$$

Hence, $\nu(E) = \int_E\ f + \epsilon\chi_{E'}\ d\mu$ for $E \in \mathcal{M}$, contradicting the maximality of f.