Chapter 6. L^p space and Banach space

Throughout this chapter, we assume $(X, \mathcal{M}, \mu)$ is a measure space. For $1 \leq p < \infty$, we define

$$L^p(X, d\mu) = \{f \ : \ \|f\|_p < \infty\}, \quad \|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$$

Theorem. (6.1.3: The L^p spaces are Banach spaces)

The $L^p(X, d\mu)$, $1 \leq p < \infty$, are complete normed linear spaces (Banach spaces), that is,

- $\|f\|_p \geq 0, \quad \|f\|_p = 0 \ \Leftrightarrow \ f = 0.$ *($f = 0$ means $f = 0$ a.e)*
- $\|\lambda f\|_p = |\lambda| \ \|f\|_p$ *for* $\lambda \in \mathbb{R}$ *(or $\lambda \in \mathbb{C}$)*
- $\|f + g\|_p \leq \|f\|_p + \|g\|_p$
- *If $\{f_n\} \subset L^p$ is a Cauchy sequence, then $\exists \ f \in L^p$ such that $f_n \to f$ in L^p-sense .*

The L^p spaces are Banach spaces

Let us begin with proving that L^p is a normed linear space.

- From the definition of $\|f\|_p$, it is easy to see that $\|f\|_p \geq 0$ and $\|\lambda f\|_p = |\lambda| \, \|f\|_p$
- Since $\|f\|_p = 0 \Leftrightarrow |f|^p = 0$ *a.e.*, we conclude $\|f\|_p = 0 \Leftrightarrow f = 0$ *a.e.*
- Clearly, $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$.
- It remains to prove the triangle inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ for $1 < p < \infty$.
    - We first prove **Hölder inequality**

$$\|fg\|_1 \leq \|f\|_p \|g\|_{\frac{p}{p-1}} \qquad (1 < p < \infty)$$

    - Based on Hölder inequality, we will prove **Minkowski inequality**

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Theorem. (6.1.1: Hölder inequality)

For $1 < p < \infty$, *we have*

$$\|fg\|_1 \leq \|f\|_p \|g\|_{\frac{p}{p-1}} \qquad \text{for all } f \in L^p \text{ and } g \in L^{\frac{p}{p-1}}$$

The equality $\|fg\|_1 = \|f\|_p \|g\|_{\frac{p}{p-1}}$ *holds iff* f *and* g *are scalar multiples of one another.*

1. In the general case, we may rescale f and g so that $\|f\|_p = 1 = \|g\|_{\frac{p}{p-1}}$. It suffices to prove $\|fg\|_1 \leq 1$ (Why?)
2. Denote $q = \frac{p}{p-1}$. By the concavity of log,

$$\log|fg| = \frac{1}{p}\log|f|^p + \frac{p-1}{p}\log|g|^q \leq \log\left(\frac{1}{p}|f|^p + \frac{p-1}{p}|g|^q\right)$$

3. Hence, $\|fg\|_1 \leq \left\| \frac{1}{p}|f|^p + \frac{p}{p-1}|g|^q \right\|_1 = \frac{1}{p}\|f\|_p^p + \frac{p-1}{p}\|g\|_q^q = \frac{1}{p} + \frac{p-1}{p} = 1.$

Theorem. (6.1.2: Minkowski inequality)

For $1 \leq p < \infty$, *we have*

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p \quad \text{for all } f, g \in L^p$$

Proof. Let $1 < p < \infty$.

1. $|f+g|^p = |f+g||f+g|^{p-1} \leq (|f|+|g|)\ |f+g|^{p-1}$.
2. $\|f+g\|_p^p = \left\| |f|\ |f+g|^{p-1} \right\|_1 + \left\| |g|\ |f+g|^{p-1} \right\|_1$.
3. By Hölder inequality,
   $\left\| |f| |f+g|^{p-1} \right\|_1 \leq \|f\|_p\ \| |f+g|^{p-1} \|_{\frac{p}{p-1}}$.
4. Note: $\| |f+g|^{p-1} \|_{\frac{p}{p-1}} = \left(\int |f+g|^p d\mu \right)^{\frac{p-1}{p}} = \|f+g\|_p^{p-1}$.
5. From 2-4, we have

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\ \|f+g\|_p^{p-1}$$

L^p is closed (or complete) w.r.t. $\|\cdot\|$.

Proof.

1. Let $\{f_n\}$ be a Cauchy sequence in L^p.
2. Choose a subsequence $\{f_{n_j}\}$ so that $\|f_{n_{j+1}} - f_{n_j}\|_p \leq 2^{-j}$.
3. Put $G_k = \sum_{j=1}^{k} |f_{n_{j+1}} - f_{n_j}|$.
4. By Minkowski inequality, $\|G_k\| \leq 1$ for all k.
5. Since $G_k \nearrow$, we can define $G = \lim_{k\to\infty} G_k$. By Monotone Convergence Theorem,

$$\int |G|^p \, d\mu = \lim_{k\to\infty} \int |G_k|^p \, d\mu \quad \& \quad |G| < \infty \ a.e.$$

6. $\therefore$ $\sum_{j=1}^{\infty}(f_{n_{j+1}} - f_{n_j})$ **converges a.e.**
7. $\therefore$ **The limit** $f = \lim_{j\to\infty} f_{n_j}$ **exists.** Why?
   $f_{n_1} + \lim_{k\to\infty} \sum_{j=1}^{k}(f_{n_{j+1}} - f_{n_j}) = \lim_{k\to\infty} f_{n_k}$

8. $f \in L^p$ since $\|f\| \leq \|f - f_{n_k}\|_p + \|f_{n_k}\|_p$ and

$$\|f - f_{n_k}\|_p = \|\sum_{j=k}^{\infty}(f_{n_{j+1}} - f_{n_j})\|_p < 2^{-k+1}$$

9. Moreover, we have L^p-convergence:
$\|f - f_{n_k}\|_p = \|\sum_{j=k}^{\infty}(f_{n_{j+1}} - f_{n_j})\|_p < 2^{-k+1} \to 0.$
10. It is easy to prove $\|f - f_k\| \to 0$. Why?
$\|f - f_k\| \leq \|f - f_{n_j}\| + \|f_{n_j} - f_k\| \to 0$ as $k, n_j \to \infty$.

Theorem. ($L^\infty(X, d\mu)$ is a Banach space.)

Define

$$\|f\|_\infty := \sup\{a \ : \ \mu(\{|f| \geq a\}) \neq 0\}$$

The space $L^\infty(X, d\mu) := \{f \ : \ \|f\|_\infty < \infty\}$ *is a Banach space.*

- By the definition of $\|f\|_\infty$, $Z := \{x \in X \ : \ |f(x)| > \|f\|_\infty\}$ has measure zero.
- Let $E = X \setminus Z$.
- Since $f = f\chi_E \ \ a.e.$ and $\|f\|_\infty = \sup |f\chi_E|$, it is easy to see $\|\cdot\|_\infty$ is a norm.
- Suppose $\{f_n\}$ is a Cauchy sequence in L^∞, that is, $\|f_n - f_m\|_\infty \to 0$ as $n, m \to \infty$. Then there exist Z such that $\mu(Z) = 0$ and

$$\sup_{X \setminus Z} |f_n - f_m| \ \to 0 \quad \text{as } n, m \to \infty$$

Hence, f_n converges pointwise to some f in $X \setminus Z$ and $f \in L^\infty$.

Theorem. (6.1.4: Generalized Hölder inequality)

If $1 \le p < q < r < \infty$, *then* $L^p \cap L^r \subset L^q$ *and*

$$\|f\|_q \le \|f\|_p^{\lambda} \, \|f\|_r^{1-\lambda}, \quad \frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}, \;\; 0 < \lambda < 1$$

Proof. Set $g = |f|^q$. When $a = \frac{p}{\lambda q}$ & $b = \frac{r}{(1-\lambda)q}$, $\frac{1}{a} + \frac{1}{b} = 1.$
From Hölder inequality,

$$\|f\|_q^q = \|g\|_1 \;\;=\;\; \|g\|_a \, \|g\|_b$$

6.2 Duality and Radon-Nikodym Theorem

Definition.

Let $\mathcal{B} = L^p(X, d\mu)$, $1 \le p < \infty$. We knew that $\mathcal{B}$ is a Banach space with norm $\| \cdot \| = \| \cdot \|_p$.

- *A bounded linear functional on $\mathcal{B}$ is a mapping $\ell : \mathcal{B} \longrightarrow \mathbb{R}$ such that $|\ell(f)| \le C\|f\|$ for all $f \in \mathcal{B}$.*
- *We denote by $\mathcal{B}^*$ the space of linear functionals on $\mathcal{B}$.*
- *Define $\|\ell\|_{\mathcal{B}^*} = \|\ell\|_* = \left\{ \frac{|\ell(f)|}{\|f\|} \ : \ f \in \mathcal{B} \ \& \ \|f\| \ne 0 \right\}$*

It is easy to prove that $\mathcal{B}^*$ is a normed vector space, that is,

- $\|\ell\|_* \ge 0$ and $\|\ell\| = 0 \Leftrightarrow \ell = 0$. Here, $\ell = 0$ means that $\ell(f) = 0$ for all $f \in \mathcal{B}$.
- $\|\lambda\ell\|_* = |\lambda| \ \|\ell\|_*$ & $\|\ell_1 + \ell_2\|_* = \|\ell_1\|_* + \|\ell_2\|_*$.

Dual & Isometry: $L^{\frac{p}{p-1}} = (L^p)^*$

Theorem. (6.2.2: Isometric injection : $L^{\frac{p}{p-1}} \hookrightarrow (L^p)^*$)

Let $1 < p < \infty$ and $q = \frac{p}{p-1}$. For $g \in L^q$, define $\ell_g : \mathcal{B} \to \mathbb{R}$ by

$$\ell_g(f) = \int g\ f\ d\mu, \quad \text{for } f \in L^p = \mathcal{B}.$$

Then $\ell_g \in \mathcal{B}^*$ and $\|\ell_g\|_* = \|g\|_q$. This means that the injection $L^q \hookrightarrow (L^p)^*$ defined by $g \mapsto \ell_g$ is an ***isometric injection*** *of L^q into $(L^p)^*$.*

Proof.

1. Clearly, ℓ_g is linear on $\mathcal{B}$.
2. According to Hölder inequality,

$$|\ell_g(f)| = \|fg\|_1 \leq \|f\|_p \|g\|_{\frac{p}{p-1}} \quad \text{for all } \ f \in L^p = \mathcal{B}$$

3. From the definition of $\|\cdot\|_*$ and 2, $\quad \|\ell_g\|_* \leq \|g\|_q$.

Proof of isometric injection : $L^{\frac{p}{p-1}} \hookrightarrow (L^p)^*$

4. If $g \neq 0$ and

$$f = \frac{|g|^{p-1} sgn(g)}{\|g\|_q^{q-1}} \quad \text{where} \quad sgn(g) = \frac{g}{|g|}$$

   then

$$\ell_g(f) = \|g\|_q$$

5. From 4, $\|\ell_g\|_* \geq \|g\|_q$
6. From 3 and 5, $\|\ell_g\|_* = \|g\|_q$
7. Hence, $g \mapsto \ell_g$ is an **isometric injection** of L^q into $(L^p)^*$.

♣ ♣ ♣ ♣

Question : What about $(L^p)^* \hookrightarrow L^q$? YES! See the next Theorem 6.2.3.

Theorem. (6.2.3: Representation combining Radon-Nikodym)

Let X be a metric space and let μ be a Borel measure with $\mu(X) < \infty$. For $1 < p < \infty$ and $\ell \in (L^p(X, d\mu))^*$, there exist $g \in L^{\frac{p}{p-1}}$ such that

$$\ell(f) = \int fg \, d\mu \quad \text{for all } f \in L^p$$

- **SUPER important!** This idea with $p = 2$ provides the key concept of Lax-Milgram (uniqueness and existence of PDE, Finite Element Method, and so on).
- To understand this theorem completely, we need to study Hahn-decomposition, signed measure, absolute continuity, Lebesgue differentiation, etc. However, it is a disaster that many recent mathematicians do not know its usefulness and basic structure even after they mastered Real analysis. This is the main reason why I introduce the theorem without knowledge of them here to provide a rough insight.

Theorem. (6.2.3: Continue...)

♣♢♡ *For* $1 < p < \infty$ *and* $\ell \in \mathcal{B}^* = (L^p(X, d\mu))^*$, *there exist* $g \in L^{\frac{p}{p-1}}$ *such that* $\ell(f) = \int \; f g \; d\mu$ *for all* $f \in L^p = \mathcal{B}$

Sketch of the proof. Let us understand its structure roughly.

- Denote $B_\epsilon(x) = \{y \in X \;:\; |y - x| < \epsilon\}$. Due to absolute continuity of ♠ , we can define Radon-Nikodym derivative in some sense

$$g(x) := \lim_{\epsilon \to 0} \frac{\ell(\chi_{B_\epsilon(x)})}{\mu(B_\epsilon(x))}, \quad x \in X$$

- Hence, $\ell(\chi_E) = \int_E g \; d\mu$ for any measurable set E.
- Hence, $\ell(\phi) = \int g \; \phi \; d\mu$ for any $\phi \in \mathcal{S}_{imple}$.
- The result follows from the facts that $\mathcal{S}_{imple}$ is dense in L^p and $\ell : L^p \to \mathbb{R}$ is bounded and linear.

6.3 Distribution functions

Given a real valued function f on a measure space $(X, \mathcal{M}, \mu)$ its distribution function $\lambda_f : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}$ by

$$\lambda_f(\alpha) \;=\; \mu(\{x : \; |f(x)| > \alpha\})$$

Theorem. (6.3.2: Prove it for $\mathcal{S}_{imple}$. EASY!)

Suppose that $\lambda_f(\alpha)$ is finite for all $\alpha \in \mathbb{R}^+$. For any continuously increasing function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\int \eta(|f|) d\mu = -\int \eta(\alpha) \; d\lambda_f(\alpha).$$

In particular, for $1 \le p < \infty$,

$$\int |f|^p d\mu = -\int_0^\infty \alpha^p \; d\lambda_f(\alpha) = p \int_0^\infty \alpha^{p-1} \; \lambda_f(\alpha) \; d\alpha$$