Chapter 3. Measurable functions

- A function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable if $f^{-1}(U)$ is Lebesgue measurable for every open set U.
- Let X be a metric space and let $(X, \mathcal{M}, \mu)$ be a measure space. **A function $f : X \to \mathbb{R}$ is measurable if $f^{-1}(U) \in \mathcal{M}$ whenever U is an open or closed interval, or open ray (a, ∞).** It is a simple exercise to show the followings:
  - $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$.
  - $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$.
  - $f^{-1}(E^c) = \left[f^{-1}(E)\right]^c$.
- In particular, **$f : X \to \mathbb{R}$ is measurable if $\{x \in X \ : \ f(x) > a\} \in \mathcal{M}$ for all $a \in \mathbb{R}$.**
- Given two function f and g we define

$$f \vee g = \max\{f, g\} \qquad f \wedge g = \min\{f, g\}$$
$$f^+ = f \vee 0 \qquad f^- = (-f) \vee 0$$

Proposition. (3.1.2)

If f and g are measurable, then so are
$f+g$, fg, $f \vee g$, $f \wedge g$, f^+, f^-, *and* $|f|$.

Proof. We will denote $\{f > a\} := \{x \in X : \ f(x) > a\}$

- $f+g$ is measurable because
  $\forall a \in \mathbb{R}, \quad \{f+g > a\} = \cup_{t \in \mathbb{Q}} (\{f > t\} \cap \{g > a-t\})$.
  $\mathbb{Q} :=$ the set of rational numbers.
- f^2 is measurable since $\{f^2 > a\} = X$ if $a < 0$ and
  $\forall a \geq 0, \quad \{f^2 > a\} = \{f > \sqrt{a}\} \cup \{f < -\sqrt{a}\}$.
- fg is measurable because $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$.
- f^+ is measurable because $\{f^+ > a\} = X$ if $a < 0$ &
  $\{f^+ > a\} = \{f > a\}$ if $a \geq 0$.
- $|f|$ is measurable because $|f| = f^+ + f^-$.
- $f \vee g$, $f \wedge g$ are measurable because
  $f \vee g = \frac{f+g+|f-g|}{2}, \quad f \wedge g = \frac{f+g-|f-g|}{2}$.

Theorem. (3.1.3)

If $\{f_j\}$ is a sequence of measurable functions, then $\limsup_j f_j$, $\limsup_j f_j$ are measurable.

Proof. Denote $\phi := \limsup_j f_j$.

1. Recall $\phi := \limsup_j f_j = \lim_{n\to\infty} g_n$ where $g_n = \sup_{j\geq n} f_j$.
2. $\{g_n > a\} = \cup_{j\geq n}\{f_j > a\}$. Hence, g_n is measurable.
3. Since $g_n \searrow$, $\limsup_j f_j = \inf_{n\geq 0}\ g_n$.
4. Hence, $\{\phi > a\} = \cap_{n=1}^{\infty}\{g_n > a\}$.
5. Therefore ϕ is measurable.
6. A similar proof shows that $\liminf_j f_j$ is measurable.

3.2 Integration of non-negative functions

Let $(X, \mathcal{M}, \mu)$ be measure space where X is a metric space. **If your mathematical background is poor, you regard X as $X = \mathbb{R}^2$ and μ as the standard Lebesgue measure, that is, $\mu(A) =$ the area of A. Throughout this lecture, E, E_j are a measurable set.**

- The **characteristic function of** E denoted by χ_E is the function defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

- A **simple function** is a finite linear combination of characteristic functions

$$\phi = \sum_{j=1}^{n} c_j \chi_{E_j}$$

Hence, $E_j = \{\phi = c_j\}$.

Theorem. (3.2.1)

Let $f : X \to \mathbb{R}$ be measurable and $f \geq 0$. Then

$$\phi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_{n,k}} \ + \ 2^n\chi_{F_n} \quad \nearrow \quad f$$

where $E_{n,k} = f^{-1}((k2^{-n}, (k+1)2^{-n}])$, $F_n = f^{-1}((2^n, \infty])$.
Moreover, each ϕ_n *satisfies*

$$\phi_n \leq \phi_{n+1} \quad \& \quad 0 \leq f(x) - \phi_n(x) \leq 2^{-n} \qquad \text{for } x \in X \setminus F_n$$

Proof. Straightforward.
From the above theorem, we can prove that for any measurable function f there is a sequence of simple functions ϕ_n such that $\phi_n \to f$ on any set on which f is bounded.
Why? $f = f^+ - f^-$ where $f = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

- **Definition of Lebesgue Integral for simple functions** The integral of a measurable simple function $\phi = \sum_{j=1}^{n} c_j \chi_{E_j}$ is defined to be
$$\int \phi d\mu = \sum_{j=1}^{n} c_j \ \mu(E_j)$$
- We use the convention that $0 \cdot \infty = 0$.
- If ϕ is a simple function, then $\phi \geq 0 \implies \int \phi d\mu \geq 0$.
- **Let $\mathcal{S}_{imple}$ be a vector space of measurable simple functions.** Then the integral $\int \square d\mu$ can be viewed as a linear functional on $\mathcal{S}_{imple}$, that is, $\int \square \ d\mu : \mathcal{S}_{imple} \to \mathbb{R}$ is linear.

Lemma. (3.2.2)

Let $(X, \mathcal{M}, \mu)$ be a measure space. Given a non-negative, measurable simple function ϕ and $A \in \mathcal{M}$, define

$$\nu(A) = \int_A \phi \, d\mu = \int_X \phi \chi_A \, d\mu$$

Then $(X, \mathcal{M}, \nu)$ is also a measure space.

Proof. Let $\phi = \sum_{k=1}^n c_k \chi_{E_k}$ where $E_k \in \mathcal{M}$. Assume $A = \cup_j A_j$ where $A_j \in \mathcal{M}$ are mutually disjoint. Then

$$\begin{aligned}
\nu(A) &= \int \phi \chi_A \, d\mu = \sum_{k=1}^n \int c_k \chi_{E_k} \chi_A d\mu = \sum_{k=1}^n \int c_k \chi_{E_k \cap A} d\mu \\
&= \sum_{k=1}^n c_k \mu(E_k \cap A) = \sum_{k=1}^n \sum_{j=1}^\infty c_k \mu(E_k \cap A_j) \\
&= \sum_{j=1}^\infty \sum_{k=1}^n c_k \mu(E_k \cap A_j) = \sum_{j=1}^\infty \int_{A_j} \phi d\mu = \sum_j \nu(A_j)
\end{aligned}$$

Definition. (Lebesgue integral of non-negative measurable function)

The integral of a non-negative measurable function f is defined by

$$\int f\ d\mu = \sup\left\{\int \phi d\mu\ :\ \phi \leq f\ \ \&\ \ \phi \in \mathcal{S}_{imple}\right\}$$

Recall that the integral of a measurable simple function $\phi = \sum_{j=1}^{n} c_j \chi_{E_j}$ is defined to be

$$\int \phi d\mu = \sum_{j=1}^{n} c_j\ \mu(E_j)$$

From the definition, we obtain

$$f \leq g \quad \Longrightarrow \quad \int f d\mu \leq \int g d\mu$$

Theorem. (3.2.3: MCT(Monotone Convergence Thm))

If $\{f_n\}$ is a nondecreasing sequence of non-negative measurable functions, then

$$\int \lim_n f_n \, d\mu \quad = \quad \lim_n \int f_n \, d\mu$$

- Since $f_n \nearrow$, $\lim_n f_n = {}^{\exists} f$ and is measurable. Note that it is possible that $f(x) = \infty$ at some x.
- Since $\int f_n d\mu \nearrow$ and $f_n \leq f$, $\int f_n \leq \int f$ and therefore

$$\lim_n \int f_n d\mu \quad \leq \quad \int f \quad d\mu$$

- It remains to prove $\lim_n \int f_n d\mu \quad \geq \quad \int f \quad d\mu$.

Theorem. (3.2.3: Continue ......MCT)

If $f_n \nearrow$, then $\int \lim_n f_n \, d\mu \quad = \quad \lim_n \int f_n \, d\mu$

Continue... Aim to prove $\lim_n \int f_n d\mu \quad \geq \quad \int f \quad d\mu$.

- Since $\int f \; d\mu = \sup\{\int \phi d\mu \; : \; \phi \leq f, \quad \phi \in \mathcal{S}_{imple}\}$, it suffices to prove that for any α, $0 < \alpha < 1$ and any $\phi \in \mathcal{S}_{imple}$ with $\phi \leq f$,

$$\lim_n \int f_n d\mu \quad \geq \quad \alpha \int \phi \quad d\mu$$

- Let $E_n = \{f_n \geq \alpha\phi\}$. Then

$$\int f_n d\mu \;\geq\; \int_{E_n} f_n d\mu \;\geq\; \int_{E_n} \alpha\phi \; d\mu \;:\overset{define}{=}\; \alpha \; \nu(E_n)$$

- Since ν is a measure and $E_n \nearrow X$, $\lim_n \nu(E_n) = \nu(X) = \int \phi d\mu$. Thus, $\lim_n \int f_n d\mu \geq \alpha \int \phi d\mu$.

Corollary. (3.2.4: $\phi_n \nearrow f$)

Let $\mathcal{M}^+_{able}$ be the set of non-negative measurable functions.

- *If $\phi_n \in \mathcal{S}_{imple}$ and $\phi_n \nearrow f$ for some $f \in \mathcal{M}_{able}$, then*

$$\lim_n \int \phi_n d\mu \quad = \quad \int f d\mu$$

- *The map $\int \Box d\mu \; : \; \mathcal{M}^+_{able} \to \mathbb{R}$ is linear.*

Proof. Let $f, g \in \mathcal{M}^+_{able}$, $\quad \phi \in \mathcal{S}^+_{imple} \nearrow f$, and $\psi \in \mathcal{S}^+_{imple} \nearrow g$. Then

- $\int f + g \; d\mu = \int \lim_n(\phi_n + \psi_n) \; d\mu$
  $= \lim_n \int (\phi_n + \psi_n) \; d\mu = \int f \; d\mu + \int g \; d\mu$
- $\int \alpha f \; d\mu = \int \lim_n \alpha\phi_n d\mu = \alpha \lim_n \int \phi_n \; d\mu = \int f \; d\mu.$

Proposition. (3.2.7: $f = 0$ almost everywhere)

Let $f \in \mathcal{M}^+_{able}$. Then

$$\int f d\mu = 0 \iff f = 0 \ a.e.$$

Proof.

- If $f \in \mathcal{S}^+_{imple}$, then the statement is immediate.
- If $f = 0$ $a.e.$ and $\phi \le f$, then $\phi = 0$ $a.e.$, and hence $\int f = \sup\{\int \phi \ : \ \phi \le f, \ \ \phi \in \mathcal{S}^+_{imple}\} = 0$.
- Conversely, let $\int f \ d\mu = 0$. Then

$$0 = \int f \ d\mu \ \ \ge \ \ \frac{1}{n}\mu(\{f > \frac{1}{n}\}), \quad n = 1, 2, \cdots$$

$\therefore \quad f = 0 \ a.e. \qquad$ Why?

$$\mu(\{f > 0\}) = \mu\left(\cup_{n=1}^{\infty}\{f > \frac{1}{n}\}\right) \le \cup_{n=1}^{\infty}\mu(\{f > \frac{1}{n}\}) = 0.$$

Lemma. (3.2.9: Fatou's Lemma)

For any sequence $f_n \in \mathcal{M}^+_{able}$, we have

$$\int \liminf_n f_n \, d\mu \quad \leq \quad \liminf_n \int f_n \, d\mu$$

Proof.

$$\liminf_n \int f_n \, d\mu = \sup_{k \geq 1} \inf_{j \geq k} \int f_j \, d\mu \geq \sup_{k \geq 1} \int \inf_{j \geq k} f_j \, d\mu$$

Since $g_k = \inf_{j \geq k} f_j \nearrow$,

$$\sup_{k \geq 1} \int \inf_{j \geq k} f_j \, d\mu = \lim_{k \to \infty} \int \inf_{j \geq k} f_j \, d\mu = \int \lim_{k \to \infty} \inf_{j \geq k} f_j \, d\mu$$

$L^1(X, d\mu)$: Complete metric space

Definition. (Integrability)

Let $(X, \mathcal{M}, \mu)$ be measure space. A function $f : X \to \mathbb{R}$ is integrable if $f \in \mathcal{M}_{able}$ & $\int |f| d\mu < \infty$. We denote by $L^1(X, d\mu)$ the class of all integrable functions. For $f \in L^1(X, d\mu)$, we define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

Question: Prove that $L^1(X, d\mu)$ is **a complete normed space** (or Banach space) when it is equipped with the norm

$$\|f\| = \int |f| d\mu \qquad (\text{its metric}: \; d(f, g) = \|f - g\| \;)$$

To answer this question, we need to study several convergence theorems.

Proposition.

$L^1(X, d\mu)$ is a normed space equipped with the norm

$$\|f\| = \int |f| d\mu$$

This means that $L^1(X, d\mu)$ is a vector space satisfying

1. $\|f\| \geq 0, \qquad \forall f \in L^1$
2. $\|f\| = 0$ *iff* $f = 0$ *a.e.*.
3. $\|\lambda f\| = |\lambda| \|f\|, \qquad \forall f \in L^1$ *and every scaler* λ.
4. $\|f + g\| \leq \|f\| + \|g\|, \qquad \forall f, g \in L^1$

.

The proof is the straightforward.

Theorem. (Lebesgue Dominate Convergence Theorem)

Assume $\{f_n\} \subset L^1$ such $f_n \to f$ a.e. and $\exists g \in L^1$ so that $|f_n| \leq g$ a.e. for all n. Then

$$f \in L^1 \quad \& \quad \int f \, d\mu = \lim_n \int f_n \, d\mu$$

Proof. Since $g + f_n \geq 0$,

$$\int \liminf_n \, (g + f_n) \, d\mu \overset{Fatou's \quad Lemma}{\leq} \liminf_n \int (g + f_n) \, d\mu$$

Hence, $\int f \, d\mu \leq \lim_n \inf \int f_n \, d\mu$.

Applying the same argument to the sequence $g - f_n \geq 0$, we obtain

$$-\int f \, d\mu \leq \liminf_n \int (-f_n) \, d\mu = -\limsup_n \int f_n \, d\mu$$

Example. (Gaussian function)

The fundamental solution of the heat equation in 1-D is

$$G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

Then

$$\int_{\mathbb{R}} G(x,t)\, dx = 1 \quad \textit{for all } t > 0 \quad \& \lim_{t \to 0^+} G(x,t) = 0 \ \textit{a.e.}$$

- Let $f_n(x) = G(x, 1/n)$. Then $f_n \to f = 0$ *a.e.* and

$$\int f \, d\mu = 0 \;\neq\; 1 = \lim_n \int f_n \, d\mu$$

This is the reason why LDC requires the assumption that $\{f_n\}$ is **dominated** by a fixed L^1-function g.

Corollary. (3.3.2)

Let $\{f_j\} \subset L^1$ *s.t.* $\sum_j \int |f_j| \, d\mu < \infty$. *Then* $\exists\, f \in L^1$ *such that*

$$\lim_{n\to\infty} \sum_{j=1}^{n} f_j = f \ a.e. \quad \& \quad \int f \, d\mu = \sum_j \int f_j \, d\mu$$

(※ Denote $f = \sum_{j=1}^{\infty} f_j$.*)*

1. Let $g_n = \sum_{j=1}^{n} |f_j|$ and $g = \sum_{j=1}^{\infty} |f_j|$.
2. Since $g_n \nearrow g$, it follows from the monotone convergence theorem that

$$\int g d\mu = \lim_n \int g_n \, d\mu = \sum_j \int |f_j| \, d\mu < \infty$$

   Hence, $g \in L^1$ and $g < \infty \ \ a.e.$
3. Since $\left|\sum_{j=1}^{n} f_j\right| < g \ a.e$ and $g \in L^1$, the result follows by the Dominate Convergence Theorem.

Theorem. (3.3.3: $\mathcal{S}_{imple}$ is dense in L^1)

- $\mathcal{S}_{imple}$ *is* **dense** *in* L^1, *i.e., every element in* L^1 *is a* L^1-*limit of a sequence of elements in* $\mathcal{S}_{imple}$.
- $C_0(\mathbb{R})$ *is* **dense** *in* $L^1(\mathbb{R}, \mathcal{M}, \mu)$, *where* μ *is any Borel measure on* $\mathbb{R}$. *Here, the definition of* $C_0(\mathbb{R})$ *is* $C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) \ : \ \exists N \ \ s.t. \ f(x) = 0 \ \ for \ |x| > N\}$.

Proof of the first statement: $\mathcal{S}_{imple}$ is dense in L^1.

- Let $f \in L^1$. By Thm 3.2.1,

$$\exists \ \phi_n \in \mathcal{S}_{imple} \ \ s.t. \ \ \phi_n \to f \ a.e. \ \ \& \ \ |\phi_n| < |f| \ a.e.$$

- By LDCT(Lebesgue Dominate Convergence Theorem), $\|\phi_n - f\| = \int |\phi_n - f| \ d\mu \to 0$. This completes the proof.

Proof of the second statement: $C_0(\mathbb{R})$ is **dense** in $L^1(\mathbb{R}, \mathcal{M}, \mu)$.

1. Since $\mathcal{S}_{imple}$ is **dense** in L^1, it suffices to prove that any $\phi \in \mathcal{S}_{imple} \cap L^1$ can be approximated by a sequence $\{f_n\} \subset C_0(\mathbb{R})$.
2. If $\phi = \chi_{(0,1)}$, then a sequence of continuous functions

$$f_n(x) := \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 0 < -1/n \\ 0 & \text{if } x > 1 + 1/n \\ \text{linear} & \text{otherwise} \end{cases} \quad \rightarrow \quad \phi = \chi_{(0,1)} \text{ in } L^1\text{-sense.}$$

   Indeed, $\|f_n - \phi\| = 1/n \rightarrow 0$.
3. Similarly, if A is a finite union of bounded open intervals, then $\phi = \chi_A$ can be approximated by a sequence $\{f_n\} \subset C_0(\mathbb{R})$.

Continue....

4. Let E be a Borel measurable set with $\mu(E) = \|\chi_E\| < \infty$. That is,

$$\mu(E) = \inf \left\{ \sum_j \mu(I_j) \ : \ E \subset \cup I_j, \quad I_j = (a_j, b_j) \right\} \ < \ \infty$$

5. Hence, for any $\epsilon > 0$, $\exists$ a finite union of open intervals $A = \cup_{j=1}^N I_j$ such that

$$\|\chi_E - \chi_A\| = \mu(E \triangle A) < \epsilon$$

   where $E \triangle A = (E \setminus A) \cup (A \setminus E)$.

6. Since $\epsilon > 0$ is arbitrary, χ_E can be approximated by a sequence $\{f_n\} \subset C_0(\mathbb{R})$.

Theorem. (Riemann $\int$ v.s. Lebesgue $\int$)

Let $f : [0,1] \to \mathbb{R}$ *be a bounded and* **Riemann integrable**. *Then*

- $f \in L^1([0,1], d\mu)$ *where* $\mu((a,b]) = b - a$.
- *Lebesgue and Riemann integrals agrees.*

1. Let $\mathcal{P}_n = \{j2^{-n} \;:\; j = 1, \cdots, 2^n\}$, a partition of $[a,b]$.
2. Denote $E_{n,j} := (j2^{-n}, (j+1)2^{-n}]$ and

$$m_{n,j} := \inf_{x \in E_{n,j}} f(x) \quad M_{n,j} := \sup_{x \in E_{n,j}} f(x)$$
$$\phi_n = \sum_{j=1}^{2^n} m_{n,j}\, \chi_{E_{n,j}} \quad \psi_n = \sum_{j=1}^{2^n} M_{n,j} \chi_{E_{n,j}}$$

3. Therefore $\phi_n \le \phi_{n+1} \le f \le \psi_{n+1} \le \psi_n$.
4. Hence, $\exists \psi = \lim_n \psi_n$ and $\exists \psi = \lim_n \psi_n$.
5. By def'n, $L(\mathcal{P}_n, f) = \int_0^1 \phi_n(x)dx \le U(\mathcal{P}_n, f) = \int_0^1 \psi_n(x)dx$

Continue....

6. By definition of the Lebesgue integral for simple functions,

$$L(\mathcal{P}_n, f) = \int \phi_n d\mu \quad \& \quad U(\mathcal{P}_n, f) = \int \psi_n d\mu$$

7. From Riemann integrability of f,
$\inf_n U(\mathcal{P}_n, f) = \sup_n L(\mathcal{P}_n, f) \;\; = \int_0^1 f(x)\, dx$
8. By LDCT,

$$\begin{aligned} \int \phi d\mu &= \lim_n \int \phi_n \, d\mu = \lim_n U(\mathcal{P}_n, f) \\ &= \inf_n U(\mathcal{P}_n, f) = \sup_n L(\mathcal{P}_n, f) = \lim_n L(\mathcal{P}_n, f) \\ &= \lim_n \int \psi_n \, d\mu = \int \psi d\mu \end{aligned}$$

9. Hence, $\int \phi d\mu = \int_0^1 f dx = \int \phi d\mu$
10. Therefore, $\psi = f = \phi$ *a.e.*.

Theorem. (3.4.3)

Let $f(x,t) : X \times [a,b] \to \mathbb{R}$ be a mapping. Suppose that f is differentiable with respect to t and that

$$g(x) := \sup_{t \in [a,b]} \left| \frac{\partial}{\partial t} f(x,t) \right| \in L^1(X, d\mu)$$

Then $F(t) = \int f(x,t) d\mu$ is differentiable on $a \le t \le b$ and

$$\frac{\partial}{\partial t} \int f(x,t) d\mu = \int \frac{\partial}{\partial t} f(x,t) d\mu$$

For each $t \in (a,b)$, we can apply LDCT to the sequence

$$h_n(x) = \frac{f(x,t_n) - f(x,t)}{t_n - t}, \quad t_n \to t$$

($\because \quad |h_n| \le g$ from the mean value theorem.)