PARTIAL DIFFERENTIAL EQUATION TEST (2014.6.13. 1:00-5:00)

This is final-term exam for applied partial differential equations. You must prove them explicitly. Unjustified or inadequately justified answers will receive no credit.

(1) **Lax-Milgram and Ritz theory** Assume the followings:
- X is a real Hilbert space with norm $\|\cdot\|$.
- $\{X_n\}$ is sequence of finite dimensional subspace of X such that $X_n \subset X_{n+1}$ and

$$\overline{\cup_{n=1}^{\infty} X_n} = X.$$

- $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ is bilinear and symmetric and

$$|a(u,v)| \leq 5\|u\|\|v\|, \quad \frac{1}{6}\|u\|^2 \leq a(u,u), \quad \forall u, v \in X.$$

- $b : X \to \mathbb{R}$ is a bounded linear map such that $|b(u)| \leq 7\|u\|$.

Under the above assumptions, answer the following questions.

(a) There exists a unique solution $u_* \in X$ of the minimization problem

$$Minimize \quad \Phi(u) := \frac{1}{2}\, a(u,u) - b(u), \quad u \in X.$$

(b) The solution u_* of the minimization problem is the solution of variational problem:

$$a(u,\phi) = b(\phi), \quad \forall \phi \in X.$$

(c) Define the new inner product and norm

$$\langle u, v\rangle_a := a(u,v), \quad \|u\|_a := \sqrt{a(u,u)} \qquad (\forall\ u, v \in X)$$

Assume that $V := \{w \in X : b(w) = 0\}$ is nonempty set. Let $u \in V^{\perp} := \{w \in X\ :\ a(w,v) = 0.\ \forall\ v \in V\}$. Show that

$$u_* = \frac{1}{a(u,u)} u.$$

(d) Assume $u_n \in X_n$ is the solution of the variational problem

$$a(u_n, v) = b(v), \quad \forall v \in X_n.$$

Prove that

$$\|u_* - u_n\| \leq 30 \min_{v \in X_n} \|u_* - v\|$$

(e) Let $\Omega = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$. Let $X = H_0^1(\Omega)$ be the Sobolev space equipped with the norm $\|u\| = \sqrt{\int_\Omega |\nabla u|^2 + |u|^2}$. Prove that there exists a unique solution $u_* \in H_0^1(\Omega)$ of the minimization problem:

$$\min \quad \Phi(u) := \frac{1}{2}\int_\Omega |\nabla v(x)|^2 dx - \int_\Omega f(x)u(x)dx, \qquad u \in X = H_0^1(\Omega).$$

(If you use **Poincaré inequality, prove it**!) Prove that the minimizer u_* is a solution of Poisson's equation $-\nabla^2 u = f$ in Ω.

(2) **Helmholtz Decomposition** Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Prove that every vector field $F(\mathbf{r}) = (F_1(\mathbf{r}), F_2(\mathbf{r}), F_3(\mathbf{r})) \in [L^2(\Omega)]^3$ can be decomposed into

$$F(\mathbf{r}) = -\nabla u(\mathbf{r}) + \nabla \times A(\mathbf{r}) + H(\mathbf{r}) \quad \text{in } \Omega$$

where u is a scalar function, $\nabla \cdot A = 0$ and H is a vector field whose Laplacian is zero in Ω. Prove that u and A are uniquely determined up to harmonic functions;

$$u(\mathbf{r}) = -\int_\Omega \frac{\nabla \cdot F(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'$$

and

$$A(\mathbf{r}) = \int_\Omega \frac{\nabla \times F(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'.$$

(3) **Euler-Lagrange equation**

(a) Consider the curve minimization problem joining two point (a, α) and (b, β):

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) = \int_a^b \sqrt{1+(u')^2}dx \\ \text{within the set} & W = \{u \in W^{1,1}(a,b) \mid u(a) = \alpha, u(b) = \beta\} \end{array} \right.$$

If u_* is a minimizer, u_* satisfies

$$u_*''(x) = 0 \qquad a < x < b.$$

(b) Let $\Omega = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$. Consider the minimization problem

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega |\nabla u(x,y)| + |u - u_0|^2 dxdy \\ \text{within the set} & W := \{u \in W^{1,1}(\Omega) : \nu \cdot \nabla u|_{\partial\Omega} = 0\} \end{array} \right.$$

where ν is the unit outerward normal vector. If u is a minimizer, prove that u satisfies :

$$\begin{cases} \nabla \cdot (\frac{\nabla u}{|\nabla u|}) = 2(u - u_0) & \text{in} \quad \Omega \\ \nu \cdot \nabla u|_{\partial\Omega} = 0 \end{cases}$$

(c) Consider the minimal surface problem

$$\left\| \begin{array}{ll} \text{Minimize} & J(u) := \int_\Omega \sqrt{1 + u_x^2 + u_y^2} dxdy \\ \text{within the set} & W := \{u \in W^{1,1}(\Omega) : u|_{\partial\Omega} = f\} \end{array} \right.$$

If a minimizer u exist, show that u satisfies the following nonlinear problem with the Dirichlet boundary condition:

$$\begin{cases} \nabla \cdot (\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0 & \text{in} \quad \Omega \\ u|_{\partial\Omega} = f \end{cases}$$

(4) Let $u \in H^1(-1,1)$ be the weak solution of the following Dirichlet problem:

$$\begin{cases} -\frac{d}{dx}\Big(\sigma(x)\frac{d}{dx}u(x)\Big) = 0 & \text{in} \ \ (-1,1) \\ u(-1) = -2, \ u(1) = 1 \end{cases} \qquad \text{where } \sigma(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$

Show that u satisfies the transmission conditions

$$u(0^+) = u(0^-) \quad \& \quad 2u'(0^+) = u'(0^-).$$

(5) Let $\Omega := \{(r\cos\theta, r\sin\theta) \ : \ 0 < r < 1, \ 0 < \theta < \frac{2\pi}{3}\}$. Let $u = (r^{-3/2} - r^{3/2})\sin(\frac{3}{2}\theta)$ with $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$. Clearly, u satisfies

$$\nabla^2 u = 0 \ \text{ in } \ \Omega\,, \qquad u|_{\partial\Omega} = 0.$$

According to the maximum principle, $u = 0$ in Ω . What is wrong with this conclusion?