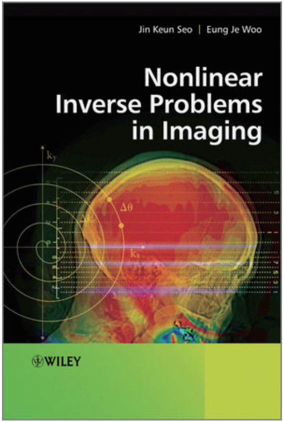
Partial Differential Equations



Part V: Regularity for second order elliptic PDE

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2nd order elliptic PDE

Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the following divergence form elliptic operator:

$$Lu = \underbrace{-\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{3} a_{ij}(x) \frac{\partial u}{\partial x_j} \right)}_{\nabla \cdot (A(x) \nabla u)} + \underbrace{\sum_{i=1}^{3} b_i(x) \frac{\partial u}{\partial x_i}}_{b(x) \cdot \nabla u} + cu(x) \tag{1}$$

where $A = (a_{ij})$ is positive definite and bounded matrix and

$$a_{ij} \in C^1(\Omega), \quad b_i, \quad c \in L^\infty(\Omega), \quad f \in L^2(\Omega)$$

The bounded linear form $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, associated with divergence form elliptic operator L is

$$a(u, \phi) = \int_\Omega \left(\sum_{j=1}^{3} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_j} \phi + cu\phi \right) dx \tag{2}$$

Interior H^2-Regularity

Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, that is,

$$a(u,\phi) = \int_\Omega f\phi dx \quad \forall \phi \in H_0^1(\Omega).$$

If $f \in L^2(\Omega)$, then the solution u has the following higher regularity: For each $V \subset\subset \Omega$, we have the estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{3}$$

the constant C depending only on V, Ω, and the coefficients of L.

The proof is very easy if you know **integration by part**.

- Since u is a weak solution of (1), we have

$$\underbrace{\int_\Omega \langle A\nabla u, \nabla\phi\rangle dx}_{=I} = \underbrace{\int_\Omega \tilde{f}\phi dx,}_{=II} \quad \forall \phi \in C_0^2(\Omega) \tag{4}$$

  where $A = (a_{ij})$ and

$$\tilde{f} = f - \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_j} - cu$$

- Choose an open set W such that $V \subset\subset W \subset\subset \Omega$. Take a cut-off function $\zeta \in C_0^1(W)$ with $\zeta|_V = 1$.

- Substitute $\phi = u\zeta^2$ into (4), we have

$$\underbrace{\int_\Omega \langle A\nabla u, \nabla u\rangle \zeta^2 dx - 2\int_\Omega \langle A\nabla u, \nabla \zeta\rangle u\zeta dx}_{=I} = \underbrace{\int_\Omega \tilde{f} u\zeta^2 dx}_{=II}$$

Observe that

$\|\nabla u\zeta\|^2_{L^2(\Omega)} \leq C_1 \int_\Omega \langle A\nabla u, \nabla u\rangle \zeta^2 dx$ where C_1 depends on the ellipticity of A.
Using $ab < \epsilon a^2 + \frac{1}{\epsilon} b^2$, we have

$$\left| \int_\Omega \langle A\nabla u, \nabla \zeta\rangle u\zeta dx \right| \leq \epsilon \|\nabla u\zeta\|^2_{L^2(\Omega)} + C_{\epsilon,1}\|u\|^2_{L^2(\Omega)}$$

where $C_{\epsilon,1}$ depends on ϵ and $\|\nabla\zeta\|_{L^\infty(\Omega)}$ and A.
Similarly,

$$\left| \int_\Omega \tilde{f} u\zeta^2 dx \right| \leq \epsilon \|\nabla u\zeta\|^2_{L^2(\Omega)} + C_{\epsilon,2}(\|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)})$$

where $C_{\epsilon,2}$ depends on ϵ and $\|b\|_{L^\infty(\Omega)}$ and $\|c\|_{L^\infty(\Omega)}$.

- By taking $\epsilon < \frac{1}{10(1+C_1)}$, all the above estimates lead to the following estimate

$$\|\nabla u\zeta\|^2_{L^2(\Omega)} \leq C_2(\|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)})$$

or

$$\|\nabla u\|^2_{L^2(V)} \leq C_2(\|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}) \tag{5}$$

- For the estimate of $\|\nabla\nabla u\|^2_{L^2(V)}$, substitute $\phi = -D_k^{-h}(D_k^h u\zeta^2)$ into (2), where $D_k^h u$ denotes the different quotient

$$D_k^h u(x) = \frac{u(x+h\mathbf{e}_k) - u(x)}{h} \approx \frac{\partial u}{\partial x_k}$$

where h is very very small and $0 < h < \ dist(W, \partial\Omega)$. Then

$$\underbrace{\int_\Omega \sum_{j=1}^{3} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j} \left(-D_k^{-h}(D_k^h u\zeta^2)\right) dx}_{=I} = \underbrace{\int_\Omega \tilde{f} \left(-D_k^{-h}(D_k^h u\zeta^2)\right) dx}_{=II}$$

- We write I as

$$I = \underbrace{\int_\Omega \sum_{j=1}^{3} a_{ij} \left(D_k^h \frac{\partial u}{\partial x_j}\right)\left(D_k^h \frac{\partial u}{\partial x_j}\right)\zeta^2 dx}_{=I_1} + \underbrace{\int_\Omega \sum_{j=1}^{3} D_k^h a_{ij} \left(\frac{\partial u}{\partial x_j}\right)\left(D_k^h \frac{\partial u}{\partial x_j}\right)\zeta^2 dx}_{=I_2}$$

- From the assumption of ellipticity,

$$C_1 \|\zeta D_k^h \nabla u\|_{L^2(\Omega)}^2 \leq I_1$$

- Using high school algebra with Holder and Poincare inequality,

$$|I_2| + |B| \leq \epsilon \|\zeta D_k^h \nabla u\|_{L^2(\Omega)}^2 + C_\epsilon \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}^2 \right)$$

  where ϵ can be chosen arbitrary small and $C_{\epsilon,2}$ depends on ϵ and $\|A\|_{C_1(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, and $\|c\|_{L^\infty(\Omega)}$.

- Then, we can have the estimate

$$\|\zeta D_k^h \nabla u\|_{L^2(\Omega)}^2 \quad \leq \quad C_* \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}^2 \right)$$

  Since this holds true for arbitrary small h, we have the estimate

$$\|\nabla \nabla u\|_{L^2(V)}^2 \quad \leq \quad C_* \left(\|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}^2 \right)$$

  Here, we may use some convergence theorem in Real Analysis. Combining this estimate with (5) with appropriate choices of ζ leads to (3).

Comments

If you feel tired, we may start with the equation

$$Lu = -\nabla^2 u = f \quad \text{in } \Omega$$

to prove that

$$\|\nabla\nabla u\|^2_{L^2(V)} \le C\left(\|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}\right)$$

With this simpler model, a much simpler computation gives the estimate (2). The estimate for the general case is essentially the same as that of this simple model. In order to understand this type of regularity theory, you first get an overall idea from the textbook within 15 minutes, and next close the book and try to obtain the estimate yourself. The key idea is

$$\int_\Omega \nabla u \cdot \nabla\left(-D_k^{-h}(D_k^h u\zeta^2)\right) dx = \int_\Omega f\left(-D_k^{-h}(D_k^h u\zeta^2)\right) dx$$

which leads to

$$\int_\Omega D_k^h \nabla u \cdot D_k^h \nabla u \,\zeta^2 dx = -\int_\Omega D_k^h \nabla u \cdot \nabla\zeta^2 \, D_k^h u dx + \int_\Omega f\left(-D_k^{-h}(D_k^h u\zeta^2)\right) dx$$

The proof follows from Schwartz inequalities.

Example: The constant C in the regularity estimate depends on dist$(\partial\Omega, V)$

Let Ω be the unit disk. Let $u^n(x,y) = r^n\cos(n\theta)$ with r being $r = \sqrt{x^2+y^2}$ and θ being $\tan\theta = y/x$. Then

$$-\nabla^2 u^n = 0 \quad \text{in } \Omega.$$

It is easy to see that

- $\|\partial_\theta^2 u^n\|^2_{L^2(\Omega)} \geq n^3, \quad \|u^n\|^2_{L^2(\Omega)} \leq 1$

Hence, the sequence $\{u^n\}$ satisfies $-\nabla^2 u^n = f = 0 \quad \text{in } \Omega$ and

$$\|\nabla\nabla u^n\|^2_{L^2(\Omega)} \to \infty, \quad \text{while} \quad \|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \leq 1$$

However, if $V \subset\subset \Omega$, there exist C, independent of n, such that

$$\|\nabla\nabla u^n\|^2_{L^2(V)} \leq C\left(\|u^n\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}\right)$$

The Elastic Wave Equation

- The development of the theory of elasticity took about two centuries, beginning with Galileo in the 1600s. The most difficult problem is to gain an understanding of the force involved in an elastic body. In 1821, Navier introduced the equations of motion for an elastic isotropic solid based on the existence assumption of attractive and repulsive forces between molecules of a body. In 1822, Cauchy (influenced by Navier's work) introduced the idea of pressure on surfaces internal to the body instead of considering intermolecular forces.
- Elastic & Plastic. In the case of solids, when a body is subjected to external forces it becomes deformed (strained), and the internal stress are generated within the body. The relation between stress and strain depends on the nature of the deformation and other external factors, such as temperature. If the deformation is such that the deformed body returns to its original state after the force that caused the deformation is removed, then the deformation is said to be *elastic*. If part of the deformation remains, the deformation is known as *plastic*.

The elastic wave equation for a homogeneous isotropic medium

Combining the equation of motion, Hook's law and the relation between strain and stress gives an equation for the displacement **u** :

$$\mu\nabla^2\mathbf{u} + (\lambda + 2\mu)\nabla(\nabla\cdot\mathbf{u}) + \rho\mathbf{f} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

- **u** is the displacement vector.
- μ is the shear modulus with $\mu = \frac{E}{2(1+\sigma)}$ and λ the Lamé coefficient with $\lambda = \frac{\sigma E}{(1-2\sigma)(1+\sigma)}$
- E is Young's modulus and σ Poisson's ratio

The result od combining the constitutive equation describing the relation between stress and strain with general continuum mechanics is the elastic wave equation. Continuum mechanics studies the deformation and motion of bodies ignoring the discrete nature of matter, and confines itself to relations among gross phenomena, neglecting the structure on a small scale.

Description of motion. Lagrangian and Eulerian of view

Let $R = (X_1, X_2, X_3)$ and $\mathbf{r} = (x_1, x_2, x_3)$ indicate the vector position corresponding to a particle in the undeformed and deformed body, respectively.

- In Lagrangian description, we follows the motion of a specified particle R by describing

$$\mathbf{r} = \mathbf{r}(R, t)$$

- In Eulerian description, we are interested in the particle that occupies a given point $\mathbf{r}$ in space at a given time by describing

$$R = R(\mathbf{r}, t)$$

provided the Jacobian determinant

$$\det(J) = \det\left(\frac{\partial(x_1, x_2, x_3)}{\partial(X_1, X_2, X_3)}\right) \neq 0, \quad J = \begin{pmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{pmatrix}$$

- Vector dR and $d\mathbf{r}$ represent the same two close points before and after deformation, respectively.

- Using Lagrangian description,

$$(ds)^2 - (dS)^2 = |d\mathbf{r}|^2 - |dR|^2 = \langle (JJ^T - I)dR, dR \rangle$$

where I is the identity matrix and $L = \frac{1}{2}(JJ^T - I)$is known as the Green finite strain tensor.

- Using Eulerian description,

$$(ds)^2 - (dS)^2 = \langle (I - \mathbf{j}\,\mathbf{j}^T)d\mathbf{r}, d\mathbf{r} \rangle, \quad \mathbf{j} = \frac{\partial(X_1, X_2, X_3)}{\partial(x_1, x_2, x_3)}$$

where $E = \frac{1}{2}(I - \mathbf{j}\,\mathbf{j}^T)$ is known as the Eulerian finite strain tensor.

The infinitesimal strain tensor

- Assuming the displacement vector $\mathbf{u} = \mathbf{u}(R, t) = \mathbf{r} - R$ is small, we introduce the infinitesimal strain tensor
$$\epsilon = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$
The assumption of small deformations has effect of making the distinction between Lagrangian and Eulerian description unnecessary.
- For small deformations, we obtain
$$(ds)^2 - (dS)^2 = |d\mathbf{r}|^2 - |dR|^2 = 2\langle\epsilon\, dR, dR\rangle$$

- From the assumption of small deformations and using $\mathbf{u} = \mathbf{r} - R$,
$$L = \frac{1}{2}\left((I + \nabla\mathbf{u})(I + \nabla\mathbf{u})^T - I\right) \approx \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$
and
$$E = \frac{1}{2}\left(I - (I - \nabla\mathbf{u})(I - \nabla\mathbf{u})^T\right) \approx \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

- Since $(ds)^2 - (dS)^2 \approx 2dS(ds - dS)$, the change in length per unit length of the original element dS is

$$\frac{ds - dS}{dS} = \langle \epsilon \frac{dR}{dS}, \frac{dR}{dS} \rangle$$

- If the line element is along the X_1 axis so that $\frac{dX_1}{dS} = 1$, $\frac{dX_2}{dS} = 0 = \frac{dX_3}{dS} = 1$, then

$$\frac{ds - dS}{dS} = \epsilon_{11}$$

- To analyze the geometric meaning of the off-diagonal elements, let us consider two line elements $dR^1 = (dS^1, 0, 0)$ and $dR^2 = (0, dS^2, 0)$, and let $d\mathbf{r}^1$ and $d\mathbf{r}^2$ be the corresponding line elements in the deformed state. Then

$$d\mathbf{r}^1 \cdot d\mathbf{r}^2 \approx (u_{12} + u_{21})\, dS^1 dS^2 \approx 2\epsilon_{12}\, ds^1 ds^2.$$

  Hence, ϵ_{12} represent one-half the change in the angle between tow line elements that were originally along the X_1 and X_2 axis.

- The strain tensor was introduced by analyzing the change in length of line elements, but it does not represent the whole effect of the deformation. The difference $d\mathbf{u} = \mathbf{u}(R + dR) - \mathbf{u}(R)$ fully describes the deformation of the vicinity of R by expanding it in a Taylor series

$$d\mathbf{u} = \mathbf{u}(R + dR) - \mathbf{u}(R) = \frac{\partial \mathbf{u}}{\partial R} dR = \epsilon dR + \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^T\right) dR$$

The Stress Tensor

Cauchy introduced the idea of pressure on surfaces internal to the body with the pressure not perpendicular to the surface which led to the concept of stress.
Let us use the Eulerian description of motion, and let $p(\mathbf{r}, t)$ indicate the value of some property of the medium (e.g., pressure, temperature, velocity) at given point $\mathbf{r}$ at time t.

- As t varies, different particle (identified by different values of R) occupy the same spatial point $\mathbf{r}$. Let us concentrate on a single particle R:

$$\mathbf{r} = \mathbf{r}(R, t), \quad p(R, t) = p(\mathbf{r}(R, t), t).$$

- When a body is in motion, the description of the time rate of change of a given property depends on how the motion is described. Assume that we are interested in measuring the time rate of change of some property such as velocity of river as a function of position and time.

- We can do at least two things.
  - One is to conduct the measurement at a point $\mathbf{r}$ that remains fixed to the shore line. The local time rate of change obtained in this way is

$$\frac{\partial}{\partial t} p(\mathbf{r}, t)$$

  - The second thing we can do is to measure the property from a canoe (which is a representative of a particle in a continuous medium) that floats along the river. The time rate of change determined from these measurement is known as the material derivative of p:

$$\frac{D}{Dt} p(\mathbf{r}, t) = \frac{\partial}{\partial t} p(\mathbf{r}, t) + (\mathbf{v} \cdot \nabla) p(\mathbf{r}, t)$$

    where $\mathbf{v}(\mathbf{r}, t) = \frac{D\mathbf{r}}{Dt} = \frac{\partial \mathbf{r}(R, t)}{\partial t}$ represents the velocity field in the spatial description.

- In continuum mechanics two different type of forces are recognized, body force (such as gravitational force) which acts at a distance within a body or between bodies, and surface or contact forces, which only depend on the surface of contact of either two bodies in contact or any tow portions of a body separated by a imaginary surface. An example of a contact force is the hydrostatic pressure on the surface of a body immersed in a fluid.

Material derivatives

Let $p(\mathbf{r}, t)$ indicate the value of some property of the medium (e.g., pressure, temperature, velocity) at given point $\mathbf{r}$ at time t.

- $\frac{\partial}{\partial t} p(\mathbf{r}, t)$ represents the local time rate of change.
- The material derivative $\frac{D}{Dt} p(\mathbf{r}, t) = \frac{\partial}{\partial t} p(\mathbf{r}, t) + (\mathbf{v} \cdot \nabla) p(\mathbf{r}, t)$ is known as the convective time rate of change, and aries from the motion of the particles in the medium.
- Expressing $\mathbf{r}$ in terms of $\mathbf{u}$, the velocity $\mathbf{v} = \frac{\partial \mathbf{u}(R,t)}{\partial t}$ leads to

$$\mathbf{v} = \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{u}(\mathbf{r}, t)$$

- The acceleration of a particle is the material time rate of change of velocity of a particle:

$$\mathbf{a}(\mathbf{r}, t) = \frac{D}{Dt} \mathbf{v}(\mathbf{r}, t) = \frac{\partial \mathbf{v}(\mathbf{r}, t)}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v}(\mathbf{r}, t)$$

Conservation of mass and Balance of linear momentum

Let V be an arbitrary body with its surface S.

- Conservation of mass of a volume V having variable density ρ is given by

$$\frac{d}{dt}\int_V \rho dV = 0$$

- The balance of linear momentum $\mathbf{P} = \int_V \rho \mathbf{v} dV$ is

$$\frac{d}{dt}\int_V \rho \mathbf{v} dV = \int_S \mathbf{T} dS + \int_V \rho \mathbf{f}\, dV$$

  where $\mathbf{T} = \mathbf{T}(\mathbf{n}) = \frac{dF}{dS}$ is the stress vector associated to the normal $\mathbf{n}$ and $\mathbf{f}$ is the body force per unit volume. Note that the force across any infinitesimal surface element dS will be equal to $\mathbf{T}dS$.

The elastic wave equation for a homogeneous isotropic medium

In terms of the stress dyadic, the stress vector can be written as

$$\mathbf{T}(\mathbf{n}) = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{n}, \quad \boldsymbol{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{212} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}$$

Here, the diagonal element of $\boldsymbol{\tau}$ are known as normal stress, while the off-diagonal elements are known as shearing stress.

- Euler equation of motion. From the balance of linear momentum, it follows from the divergence theorem that

$$\frac{d}{dt}\int_V \rho \mathbf{v} dV = \int_S \mathbf{T} dS + \int_V \rho \mathbf{f}\, dV \quad \Rightarrow \quad \rho \frac{D\mathbf{v}}{dt} = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f}$$

- **Isotropic elastic solid**. Cauchy generalized Hook's law to elastic solids by propagating that stress and strain are linearly related. In dyadic form for an isotropic media, the law takes form:

$$\boldsymbol{\tau} = \lambda \nabla \cdot \mathbf{u}\, I + \mu \left(\nabla u + (\nabla \mathbf{u})^T \right)$$

where I is the identity matrix.

- **Under the small-deformation assumption,**

$$\frac{D\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \approx \frac{\partial \mathbf{v}}{\partial t} \approx \frac{\partial^2 \mathbf{u}}{\partial t^2}.$$

Hence, **the elastic wave equation for an isotropic medium** will be written in vector form

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla(\lambda \nabla \cdot \mathbf{u}) + \nabla \cdot \left(\mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \right) + \rho \mathbf{f}$$

For a homogeneous isotropic medium, we have

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \underbrace{\mu \nabla^2 \mathbf{u}}_{shear} + \underbrace{(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u})}_{compression} + \rho \mathbf{f}$$

Isotropic viscoelastic medium

The propagation of an acoustic wave in a locally homogeneous isotropic viscoelastic medium is governed by

$$\rho\partial_t^2\mathbf{u} = \mu\nabla^2\mathbf{u} + (\lambda + 2\mu)\nabla\nabla\cdot\mathbf{u} + \mu_v\partial_t\nabla^2\mathbf{u} + (\xi + \mu_v)\partial_t\nabla\nabla\cdot\mathbf{u}$$

where μ_v the shear viscosity accounting for attenuation within the medium, and ξ the viscosity of the compressible wave. Reference: Sinkus *et al* MR-elastography

Harmonic displacement

Assume that the subject is isotropic elastic medium. When we apply sinusoidal vibration with an angular frequency ω, 50Hz$\le \frac{\omega}{\pi} \le$200Hz through the surface of the object, it generates the internal time-harmonic displacement $\mathbf{u}$ which is dictated by the following elasticity equation:

$$\nabla\cdot((\mu + i\omega\mu_v)\nabla\mathbf{u}) + \nabla\left((\lambda + 2\mu + i\omega(\xi + \mu_v))\nabla\cdot\mathbf{u}\right) + \omega^2\rho\mathbf{u} = 0$$

We should note that even if $\nabla\cdot\mathbf{u}\approx 0$ (nearly incompressible), the quantity $\lambda\nabla\nabla\cdot\mathbf{u}$ is NOT negligible.

- According to Helmholtz decomposition, **u** can be decomposed into

$$\mathbf{u} = \mathbf{u}_L + \mathbf{u}_T + \mathbf{u}_H \qquad \text{(curl-free + div-free + harmonic)}$$

- Since the harmonic part $\mathbf{u}_H$ accounts for bulk motion, we may drop out because we are dealing with waves. Hence,

$$\begin{aligned} -\omega^2 \rho \mathbf{u}_T &= \nabla \cdot ((\mu + i\omega\mu_v)\nabla \mathbf{u}_T) \\ -\omega^2 \rho \mathbf{u}_L &= \nabla\, ((\lambda + 2\mu + i\omega(\xi + \mu_v))\nabla \cdot \mathbf{u}_L) \end{aligned}$$

- Since $\nabla \times (\nabla \times \mathbf{u}_L) = 0$, $\nabla\nabla \cdot \mathbf{u} = \nabla\nabla \cdot \mathbf{u}_L = \nabla^2 \mathbf{u}_L$. Hence, if the subject is locally homogeneous, we have

$$\nabla^2 \mathbf{u}_L = \frac{-\omega^2 \rho}{\lambda + 2\mu + i\omega(\xi + \mu_v)} \mathbf{u}_L \approx 0 \quad (\textit{why?} \quad \lambda \approx \infty)$$