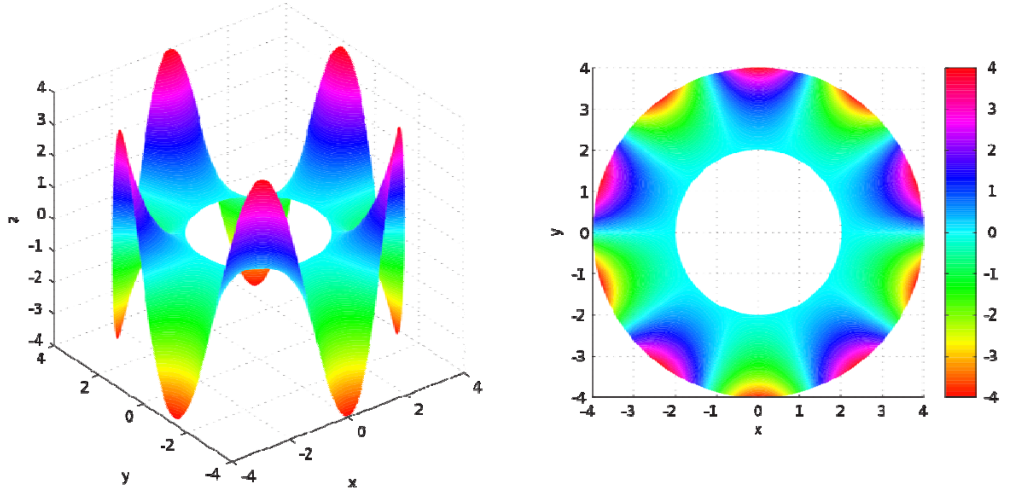
Laplace Equation

$$\nabla \cdot \nabla u = 0$$

Laplace's equation describes equilibrium solutions (or steady state or unchanging in time) such as steady-state electrical potential, steady-state temperature, and so on.



Boundary Value Problem for $\nabla^2 u = 0$

Let Ω be a domain in $\mathbb{R}^3$. The Laplace equation of a scalar variable u is

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \quad \text{in} \quad \Omega.$$

There are three types of boundary conditions:

- Dirichlet boundary condition: $u|_{\partial\Omega} = f$ on the boundary $\partial\Omega$.
- Neumann boundary condition: $\nabla u \cdot \mathbf{n}|_{\partial\Omega} = g$ on the boundary $\partial\Omega$.
- Mixed boundary condition: for two non-overlapping parts Γ_D and Γ_N with $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}$,

$$u|_{\Gamma_D} = f \qquad \text{and} \qquad \frac{\partial u}{\partial \mathbf{n}}|_{\Gamma_N} = \nabla u \cdot n|_{\Gamma_N} = g.$$

When there exists an internal source ρ in Ω, u satisfies the Poisson equation:

$$-\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = \rho \quad \text{in} \quad \Omega$$

Boundary Value Problem

For a given $\rho \in C(\bar{\Omega})$ and $h \in C(\partial\Omega)$, consider the following Poisson equation with a Dirichlet boundary condition:

$$\begin{cases} -\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = \rho & \text{in} \quad \Omega \\ u|_{\partial\Omega} = f. \end{cases} \tag{1}$$

- By the principle of superposition, we can decompose its solution u as

$$u(\mathbf{r}) = v(\mathbf{r}) + h(\mathbf{r}), \qquad \mathbf{r} \in \Omega$$

where v and h satisfy

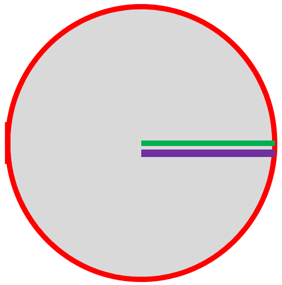
$$\begin{cases} -\nabla^2 v = \rho & \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases} \quad \text{and} \quad \begin{cases} -\nabla^2 h = 0 & \text{in } \Omega \\ h|_{\partial\Omega} = f. \end{cases}$$

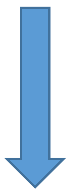
Examples

- $u = \frac{1}{2}x(x-1)$ is solution of
$$u''(x) = 1 \;\; (0 < x < 1), \qquad u(0) = u(1) = 0.$$
- $u(x,y) = \frac{x^2+y^2-1}{4}$ is the solution of
$$\nabla^2 u = 1 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = 0$$
where Ω is the unit disk centered at the origin.
- Let u be the solution of
$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega = (0,\pi)\times(0,\pi) \\ u(x,0) = g(x), \; 0 = u(x,\pi) = u(0,y) = u(\pi,y), & 0 \le x, y \le \pi \end{cases}$$
Use separable variables and Fourier analysis to get
$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh n(\pi - y) \sin nx \quad \text{where} \sum_{n=1}^{\infty} c_n \sinh n\pi \sin nx = g(x).$$

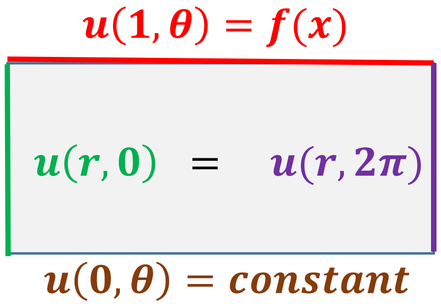
Laplace Equation in two dimension

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$





$$x = r\cos\theta$$
$$y = r\sin\theta$$



Theorem (Laplace Equation in a Circle)

Let $\Omega = B_1(0)$. Suppose u is a solution of

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. We can express u as

$$u(r\cos\theta,\ r\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\tilde{\theta}, \sin\tilde{\theta})\frac{1-r^2}{1+r^2-2r\cos(\theta-\tilde{\theta})}\,d\tilde{\theta}.$$

- For convenience, we denote $\hat{u}(r,\theta) = u(r\cos\theta,\ r\sin\theta)$. Then, $\hat{u}(r,\theta)$ satisfies

$$\frac{\partial^2 \hat{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \hat{u}}{\partial \theta^2} = 0 \quad \text{in } [0,1)\times[0,2\pi]$$

with

$$\underbrace{u(1,\theta) = f(\theta),}_{boundary\ \ condition}\ \underbrace{u(r,2\pi) = u(r,0), \quad u(0,\theta) = u(0,0)}_{continuity\ \ condition}.$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial \theta}{\partial \theta} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \quad \Longrightarrow \quad \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}} & \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{r}} \\ \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} & \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\theta}} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}$$

$$\nabla^2 = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

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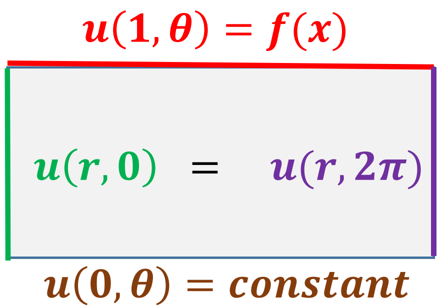
where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. We can express u as

$$u(r\cos\theta,\ r\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\tilde{\theta}, \sin\tilde{\theta})\frac{1-r^2}{1+r^2-2r\cos(\theta-\tilde{\theta})}d\tilde{\theta}.$$



Continue the proof: Separation of variables

Consider $\hat{u}(r,\theta) = v(r)w(\theta)$ satisfying

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\underbrace{v(r)w(\theta)}_{\hat{u}(r,\theta)} = 0 \quad \text{in } [0,1)\times[0,2\pi]$$

with $u(r,2\pi) = u(r,0)$, $u(0,\theta) = u(0,0)$.

- $\hat{u}(r,\theta) = v(r)w(\theta)$ must satisfy

$$\frac{r^2 v''(r) + rv'(r)}{v(r)} = -\frac{w''(\theta)}{w(\theta)} = \lambda = \text{ a constant.}$$

  This leads to

  - $w''(\theta) + \lambda w(\theta) = 0$ with $w(0) = w(2\pi)$.
  - $r^2 v'' + rv' - n^2 v = 0$ & $v(0) < \infty$

- The eigenvalue problem for w has

$$\lambda = 0 \leftrightarrow w = 1 \qquad \lambda = n^2 \ \ (n = 1, 2, \cdots) \leftrightarrow \sin n\theta,\ \cos n\theta.$$

- For each eigenvalue $\lambda = n^2$, $v(r) = r^n$.

Theorem (Laplace Equation in a Circle)

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where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. We can express u as

$$u(r\cos\theta,\ r\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\tilde{\theta}, \sin\tilde{\theta})\frac{1-r^2}{1+r^2-2r\cos(\theta-\tilde{\theta})}\,d\tilde{\theta}.$$

- $w''(\theta) + \lambda w(\theta) = 0$ with $w(0) = w(2\pi)$.
- $r^2 v'' + rv' - n^2 v = 0$ & $v(0) < \infty$
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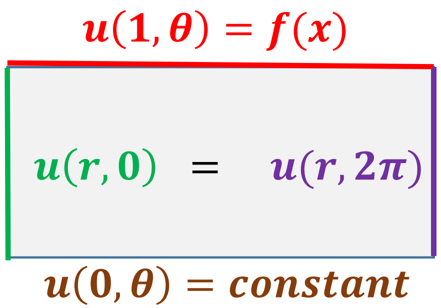

- Hence, all possible polar separable solutions are

$$v(r)w(\theta) = r^n\cos n\theta \quad \text{or} \quad r^n \sin n\theta \qquad (n = 0, 1, 2, 3, \cdots).$$

- Claim: $\hat{u}$ can be expressed as

$$\hat{u}(r,\theta) = a_0 + \sum_{n=1}^{\infty}\left(a_n r^n \cos n\theta + b_n r^n \sin n\theta\right).$$

- Since this expression for $\hat{u}$ satisfies all conditions except $\hat{u}(1,\theta) = f(\theta)$, it suffices to show that f can be expressed as

$$f(\theta) = \hat{u}(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

According to Fourier analysis, we can do it.

Theorem (Laplace Equation in a Circle)

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$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad in\ \Omega, \qquad u|_{\partial\Omega} = f$$

where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$. We can express u as

$$u(r\cos\theta,\ r\sin\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\cos\tilde{\theta}, \sin\tilde{\theta})\frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \tilde{\theta})}d\tilde{\theta}.$$

- Hence, the solution $u(r\cos\theta, r\sin\theta) = \hat{u}(r,\theta)$ can be expressed as

$$\hat{u}(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\tilde{\theta}) \underbrace{\left[\frac{1}{2} + \sum_{n=1}^{\infty} r^n(\cos n\theta\cos n\tilde{\theta} + \sin n\theta\sin n\tilde{\theta})\right]}_{=\frac{1}{2}+\sum_{n=1}^{\infty} r^n \cos n(\theta-\tilde{\theta})} d\tilde{\theta}.$$

- Denoting $z = re^{i(\theta-\tilde{\theta})}$, we have

$$\frac{1}{2} + \sum_{n=1}^{\infty} r^n\cos n(\theta - \tilde{\theta}) = \Re\left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right) = \frac{1 - r^2}{2(1 + r^2 - 2r\cos(\theta - \tilde{\theta}))}$$

and

$$\hat{u}(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\tilde{\theta})\frac{(1 - r^2)}{2(1 + r^2 - 2r\cos(\theta - \tilde{\theta}))}d\tilde{\theta}.$$

Theorem (Mean value property and maximum principle)

Assume that $\nabla^2 u(\mathbf{r}) = 0$ *in* $\Omega \subset \mathbb{R}^2$*. Here,* $\mathbf{r} = (x, y)$*. If* $\overline{B_{r_0}(\mathbf{r}_0)} \subset \Omega$*, then* $u(\mathbf{r}_0)$ *is the average of* u *over the circle* $\partial B_{r_0}(\mathbf{r}_0)$*:*

$$u(\mathbf{r}_0) = \frac{1}{2\pi r_0} \oint_{\partial B_{r_0}(\mathbf{r}^0)} u(\mathbf{r}) d\ell_{\mathbf{r}}.$$

Moreover, u *can not have a strict local maximum or minimum at any interior point of* Ω*.*

- Writing $\hat{u}(r, \theta) = u\left(\mathbf{r}_0 + r_0(r\cos\theta, r\sin\theta)\right)$, we have

$$\hat{u}(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(1, \tilde{\theta}) \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \tilde{\theta})} d\tilde{\theta}.$$

- The above identity directly yields

$$u(\mathbf{r}_0) = \hat{u}(0, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(1, \tilde{\theta}) d\tilde{\theta} = \frac{1}{2\pi r_0} \oint_{\partial B_{r_0}(\mathbf{r}_0)} u(\mathbf{r}) d\ell_{\mathbf{r}},$$

  which means that $u(\mathbf{r}_0)$ is the average of u over the circle $\partial B_{r_0}(\mathbf{r}_0)$.
- The maximum principle follows from the mean value property.

Theorem (Uniqueness of Dirichlet problem)

Suppose $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy

$$\nabla^2 u = \nabla^2 v \quad \textit{in} \quad \Omega, \qquad u|_{\partial\Omega} = v|_{\partial\Omega}.$$

Then $u = v$ in Ω.

- Denoting $w = u - v$, w satisfies

$$\nabla^2 w = 0 \quad \text{in} \quad \Omega, \qquad w|_{\partial\Omega} = 0.$$

From the maximum principle, w can not have a strict maximum or minimum inside Ω. Since $w|_{\partial\Omega} = 0$, $w = 0$ in Ω.