Chapter 4. Product spaces

Throughout this chapter, we assume that $(X_j, \mathcal{M}_j, \mu_j), j = 1, 2$ is two $\sigma-$finite measure spaces. Recall that the measure μ_j is called -finite, if X is the countable union of measurable sets of finite measure. Let $X = X_1 \times X_2$ and let $\mathcal{R} = \{E_1 \times E_2 \ : \ E_j \in \mathcal{M}_j\}$.

A product measure space $(X, \mathcal{M}, \mu)$ is constructed as follows:

- Define the pre-measure μ' on $\mathcal{R}$ by $\mu'(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2)$
- By Carathéodory's theorem, we obtain a complete measure μ on X whose σ-algebra of measurable sets contain the product algebra $\mathcal{M}_1 \otimes \mathcal{M}_2 := \sigma$-algebra generated by $\mathcal{R}$.
- Since μ_j is $\sigma-$finite, so is μ.

Theorem. (4.1.1: Fubini)

Assume $f \in L^1(d\mu)$. *Then*

$$\int_{X_2} f(x,y)\,d\mu_2(y) \in L^1(d\mu_1), \quad \int_{X_1} f(x,y)\,d\mu_1(x) \in L^1(d\mu_2)$$

and

$$\int_X f d\mu = \int_{X_1}\left[\int_{X_2} f(x,y)\,d\mu_2(y)\right] d\mu_1(x) = \int_{X_2}\left[\int_{X_1} f(x,y)\,d\mu_1(x)\right] d\mu_2(y)$$

The strategy of the proof.

1. Begin by proving the result for $f(x,y) = \chi_{E_1 \times E_2}$. It is trivial!
2. Then, prove it for $f \in \mathcal{S}_{imple}$.
3. Finally, extend it for general $f \in L^1(d\mu)$.

Differentiation Theory

5.1 Differentiation Theory of functions

Throughout this subsection, we consider a bounded function $f : [a, b] \to \mathbb{R}$. We will study **a necessary and sufficient condition that f' exist almost everywhere and**

$$f(y) - f(x) = \int_x^y f'(x) d\mu, \quad \mu((x, y)) = |y - x|.$$

- If f is Cantor function, then $f' = 0$ almost everywhere but
$$1 = f(1) - f(0) \neq 0 = \int_x^y f'(x) d\mu$$
- **Lebesgue's Theorem 5.1.1**: Every monotonic function $f : [a, b] \to \mathbb{R}$ is differentiable almost everywhere.
- Recall that the derivative of f at x exists if the following all four numbers are the same finite value:
$$\lim\ \inf_{h \to 0^\pm} \frac{f(x+h) - f(x)}{h}, \quad \lim\ \sup_{h \to 0^\pm} \frac{f(x+h) - f(x)}{h}$$

Definition. (Bounded Variations)

Let $f : [a, b] \to \mathbb{R}$ be a function. The total variation of f on $[a, x]$ is defined to be

$$T_f(a, x) = \sup_n \quad \sup_{\mathcal{P}_n} \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})|,$$

where $\mathcal{P}_n = \{a = x_0 < x_1 < \cdots < x_n = x\}$.
The class of functions of bounded variation on $[a, b]$ is denoted by $BV[a, b]$.

It is well known that the space $BV[a, b]$ is a Banach space with norm $\|f\|_{var} = T_f(a, b)$.

♣ If $f(x) = A \sin nx$, then $T_f(0, \pi) = An$.

Theorem. (5.1.3: Jordan Decomposition)

Every $f \in BV[a,b]$ can be written as two non-decreasing functions.

Proof. Let $T(x) = T_f(a,x)$. The theorem will be proved by showing $T - f$ and T are non-decreasing since $f = T - (T - f)$.

1. Let $x < y$. $T_f(x,y) = T(y) - T(x)$
2. From the definition,

   $$T(x) = T_f(a,x) \quad \text{is a montone non-decreasing function of } x$$

   since, for $x < y$, $T(y) = T(x) + T(x,y) \geq T(x)$.
3. Clearly, $|f(y) - f(x)| \leq T_f(x,y) = T(y) - T(x)$.
4. Hence, $f(y) - f(x) \leq T(y) - T(x)$.
5. Hence, $T(x) - f(x) \leq T(y) - f(y)$.

Definition. (5.1.5: Absolute continuous)

A function $f \in BV[a, b]$ is absolute continuous iff $\forall \epsilon > 0$, there exist δ such that whenever a sequence of non-overlapping subintervals $(x_j, y_j) \subset [a, b]$ satisfies $\sum_j (y_j - x_j) < \delta$, then

$$\sum_j |f(y_j) - f(x_j)| < \epsilon$$

Note that the Cantor function is not absolute continuous.

Theorem. (5.1.6: Absolute continuity)

If f' exist almost everywhere, $f' \in L^1(d\mu)$, and

$$f(x) = \int_a^x f'(x)\ d\mu, \qquad x \in (a, b]$$

then f is absolute continuous.

Proof.

We want to prove that for a given $\epsilon > 0$, there exist δ s.t. $\sum_j (y_j - x_j) < \delta \quad \Rightarrow \sum_j |f(y_j) - f(x_j)| < \epsilon$.

1. If $|f'|$ is bounded, then we choose $\delta = \frac{\epsilon}{\|f'\|_\infty}$ and

$$\sum_j |f(y_j) - f(x_j)| \leq \sum_j \int_{x_j}^{y_j} |f'| d\mu \leq C \sum_j (y_j - x_j) < \|f'\|_\infty \delta = \epsilon$$

2. If $f' \in L^1(d\mu)$ but not bounded, then we decompose

$$f' = g + h \text{ where } g \text{ is bounded and } \int |h| d\mu < \frac{\epsilon}{2}$$

This is possible because the bounded functions are dense in $L^1(d\mu)$. The results follows by choosing $\delta = \frac{\epsilon}{2\|g\|_\infty}$.

Theorem. (5.1.7: absolute + singular)

Let f be continuous and non-decreasing. Then f can be decomposed into **the sum of an absolute continuous function and a singular function,** *both monotone.*

Proof.

1. f' exist almost everywhere by Lebesgue's theorem.
2. Since f is continuous, $\int_a^x \frac{f(t+h)-f(t)}{h} d\mu = \frac{1}{h}\int_x^{x+h} f(t)d\mu - \frac{1}{h}\int_a^{a+h} f(t)d\mu \to f(x)-f(a)$ as $h \to 0$.
3. Since $\frac{f(t+h)-f(t)}{h} \to f'(t)$ $a.e.$ as $h \to 0$, by Fatou's lemma,
$$\int_a^x f'(t)d\mu \leq \liminf \int_a^x \frac{f(t+h)-f(t)}{h} d\mu = f(x) - f(a)$$
4. Since $f' \geq 0$, $f' \in L^1(d\mu)$. Set $g = \int_a^x f'(t)dt$ and $h = f - g$. Then g is absolute continuous and $h' = 0 a.e.$.