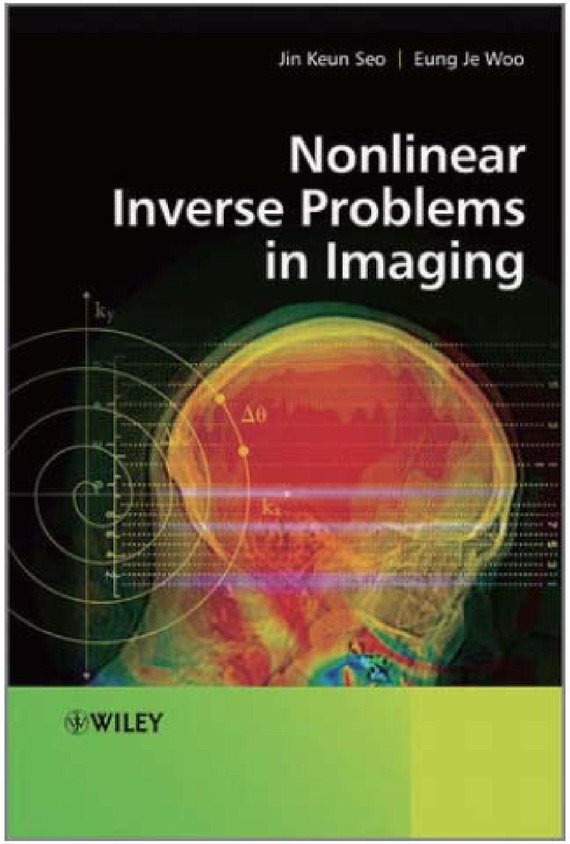

Part V: Regularity for second order elliptic PDE

Homepage

http://www.seojinkeun.com/♯!pde-course-/c1laj

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Let Ω be a bounded domain in $\mathbb{R}^3$. Consider the following divergence form elliptic operator:

$$Lu = \underbrace{-\sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3} a_{ij}(x)\frac{\partial u}{\partial x_j}\right)}_{\nabla\cdot(A(x)\nabla u)} + \underbrace{\sum_{i=1}^{3} b_i(x)\frac{\partial u}{\partial x_i}}_{b(x)\cdot\nabla u} + cu(x) \tag{1}$$

where $A = (a_{ij})$ is positive definite and bounded matrix and

$$a_{ij} \in C^1(\Omega), \quad b_i, \quad c \in L^\infty(\Omega), \quad f \in L^2(\Omega)$$

The bounded linear form $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, associated with divergence form elliptic operator L is

$$a(u,\phi) = \int_\Omega \left(\sum_{j=1}^{3} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial \phi}{\partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_j}\phi + cu\phi\right) dx \tag{2}$$

Interior H^2-Regularity

Suppose that $u \in H^1(\Omega)$ is a weak solution of $Lu = f$, that is,

$$a(u,\phi) = \int_\Omega f\phi dx \quad \forall \phi \in H_0^1(\Omega).$$

If $f \in L^2(\Omega)$, then the solution u has the following higher regularity: For each $V \subset\subset \Omega$, we have the estimate

$$\|u\|_{H^2(V)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \tag{3}$$

the constant C depending only on V, Ω, and the coefficients of L.

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the constant C depending only on V, Ω, and the coefficients of L.

The proof is very easy if you know **integration by part**.

- Since u is a weak solution of (1), we have

$$\underbrace{\int_\Omega \langle A\nabla u, \nabla\phi\rangle dx}_{=I} = \underbrace{\int_\Omega \tilde{f}\phi dx}_{=II}, \quad \forall \phi \in C_0^2(\Omega) \tag{4}$$

  where $A = (a_{ij})$ and

$$\tilde{f} = f - \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_j} - cu$$

- Choose an open set W such that $V \subset\subset W \subset\subset \Omega$. Take a cut-off function $\zeta \in C_0^1(W)$ with $\zeta|_V = 1$.

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  Observe that

  $\|\nabla u\zeta\|^2_{L^2(\Omega)} \leq C_1 \int_\Omega \langle A\nabla u, \nabla u\rangle \zeta^2 dx$ where C_1 depends on the ellipticity of A.
  Using $ab < \epsilon a^2 + \frac{1}{\epsilon} b^2$, we have

$$\left|\int_\Omega \langle A\nabla u, \nabla\zeta\rangle u\zeta dx\right| \leq \epsilon\|\nabla u\zeta\|^2_{L^2(\Omega)} + C_{\epsilon,1}\|u\|^2_{L^2(\Omega)}$$

  where $C_{\epsilon,1}$ depends on ϵ and $\|\nabla\zeta\|_{L^\infty(\Omega)}$ and A.
  Similarly,

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- By taking $\epsilon < \frac{1}{10(1+C_1)}$, all the above estimates lead to the following estimate

$$\|\nabla u\zeta\|^2_{L^2(\Omega)} \le C_2(\|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)})$$

  or

$$\|\nabla u\|^2_{L^2(V)} \le C_2(\|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}) \tag{5}$$

- For the estimate of $\|\nabla\nabla u\|^2_{L^2(V)}$, substitute $\phi = -D_k^{-h}(D_k^h u\zeta^2)$ into (2), where $D_k^h u$ denotes the different quotient

$$D_k^h u(x) = \frac{u(x + he_k) - u(x)}{h} \approx \frac{\partial u}{\partial x_k}$$

  where h is very very small and $0 < h < dist(W, \partial\Omega)$. Then

$$\underbrace{\int_\Omega \sum_{j=1}^{3} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial}{\partial x_j}\left(-D_k^{-h}(D_k^h u\zeta^2)\right) dx}_{=I} = \underbrace{\int_\Omega \tilde{f}\left(-D_k^{-h}(D_k^h u\zeta^2)\right) dx}_{=II}$$

- We write I as

$$I = \underbrace{\int_\Omega \sum_{j=1}^{3} a_{ij}\left(D_k^h \frac{\partial u}{\partial x_j}\right)\left(D_k^h \frac{\partial u}{\partial x_j}\right)\zeta^2 dx}_{=I_1} + \underbrace{\int_\Omega \sum_{j=1}^{3} D_k^h a_{ij}\left(\frac{\partial u}{\partial x_j}\right)\left(D_k^h \frac{\partial u}{\partial x_j}\right)\zeta^2 dx}_{=I_2}$$

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$$I = \underbrace{\int_\Omega \sum_{j=1}^{3} a_{ij}\left(D_k^h \frac{\partial u}{\partial x_j}\right)\left(D_k^h \frac{\partial u}{\partial x_j}\right)\zeta^2 dx}_{=I_1} + \underbrace{\int_\Omega \sum_{j=1}^{3} D_k^h a_{ij}\left(\frac{\partial u}{\partial x_j}\right)\left(D_k^h \frac{\partial u}{\partial x_j}\right)\zeta^2 dx}_{=I_2}$$

- From the assumption of ellipticity,

$$C_1 \|\zeta D_k^h \nabla u\|^2_{L^2(\Omega)} \le I_1$$

- Using high school algebra with Holder and Poincare inequality,

$$|I_2| + |B| \le \epsilon \|\zeta D_k^h \nabla u\|^2_{L^2(\Omega)} + C_\epsilon \left(\|u\|_{H^1(\Omega)} + \|f\|^2_{L^2(\Omega)} \right)$$

  where ϵ can be chosen arbitrary small and $C_{\epsilon,2}$ depends on ϵ and $\|A\|_{C_1(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, and $\|c\|_{L^\infty(\Omega)}$.

- Then, we can have the estimate

$$\|\zeta D_k^h \nabla u\|^2_{L^2(\Omega)} \le C_* \left(\|u\|_{H^1(\Omega)} + \|f\|^2_{L^2(\Omega)} \right)$$

  Since this holds true for arbitrary small h, we have the estimate

$$\|\nabla\nabla u\|^2_{L^2(V)} \le C_* \left(\|u\|_{H^1(\Omega)} + \|f\|^2_{L^2(\Omega)} \right)$$

  Here, we may use some convergence theorem in Real Analysis. Combining this estimate with (5) with appropriate choices of ζ leads to (3).

Example: The constant C in the regularity estimate depends on $\text{dist}(\partial\Omega, V)$

Let Ω be the unit disk. Let $u^n(x,y) = r^n\cos(n\theta)$ with r being $r = \sqrt{x^2+y^2}$ and θ being $\tan\theta = y/x$. Then

$$-\nabla^2 u^n = 0 \quad \text{in } \Omega.$$

It is easy to see that

- $\|\partial_\theta^2 u^n\|^2_{L^2(\Omega)} \geq n^3, \quad \|u^n\|^2_{L^2(\Omega)} \leq 1$

Hence, the sequence $\{u^n\}$ satisfies $-\nabla^2 u^n = f = 0 \quad \text{in } \ \Omega$ and

$$\|\nabla\nabla u^n\|^2_{L^2(\Omega)} \to \infty, \quad \text{while} \quad \|u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \leq 1$$

However, if $V \subset\subset \Omega$, there exist C, independent of n, such that

$$\|\nabla\nabla u^n\|^2_{L^2(V)} \ \leq C\left(\|u^n\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}\right)$$

The elastic wave equation for a homogeneous isotropic medium

Combining the equation of motion, Hook's law and the relation between strain and stress gives an equation for the displacement **u** :

$$\mu\nabla^2\mathbf{u} + (\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) + \rho\mathbf{f} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}$$

- **u** is the displacement vector.
- μ is the shear modulus with $\mu = \frac{E}{2(1+\sigma)}$ and λ the Lamé coefficient with $\lambda = \frac{\sigma E}{(1-2\sigma)(1+\sigma)}$
- E is Young's modulus and σ Poisson's ratio

The result od combining the constitutive equation describing the relation between stress and strain with general continuum mechanics is the elastic wave equation. Continuum mechanics studies the deformation and motion of bodies ignoring the discrete nature of matter, and confines itself to relations among gross phenomena, neglecting the structure on a small scale.

Green's function for a domain Ω.

Definition

Let Ω be a smooth domain in $\mathbb{R}^3$. The Green's function of the Laplace equation for the domain Ω is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} - H(\mathbf{r}, \mathbf{r}'), \qquad \mathbf{r}, \mathbf{r}' \in \Omega$$

where $H(\mathbf{r}, \mathbf{r}')$ satisfies the following: for each $\mathbf{r}' \in \Omega$,

$$\begin{cases} -\nabla^2 H(\mathbf{r}, \mathbf{r}') = 0 & \text{for } \mathbf{r} \in \Omega \\ H(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} & \text{for } \mathbf{r} \in \partial\Omega. \end{cases}$$

- For a fixed $\mathbf{r}' \in \Omega$, the Green's function satisfies the following in the sense of distribution:

$$\begin{cases} -\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') & \text{for } \mathbf{r} \in \Omega \\ G(\mathbf{r}, \mathbf{r}') = 0 & \text{for } \mathbf{r} \in \partial\Omega. \end{cases}$$

- When $\Omega = \mathbb{R}^3$, $H = 0$ and the Green's function $G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}$.

Representation of solutions by Green's function

Theorem

We can express the solution of the Poisson's equation

$$\begin{cases} -\nabla^2 u = \rho & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

as

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' \qquad (\forall \mathbf{x} \in \Omega).$$

- The proof follows from the fact that

$$\begin{cases} -\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') & \text{for } \mathbf{r} \in \Omega \\ G(\mathbf{r}, \mathbf{r}') = 0 & \text{for } \mathbf{r} \in \partial\Omega. \end{cases}$$

- However, finding Green's function can be very very difficult if the domain is not a ball, half space, and domains with very special geometry. Solving the PDE using FEM is much much easier.

Lemma

If G is a Green's function for the domain Ω, then it is symmetric:

$$G(\mathbf{x}, \mathbf{y}) = G(\mathbf{y}, \mathbf{x}) \qquad (\forall \mathbf{x}, \mathbf{y} \in \Omega)$$

where $\mathbf{x} = (x_1, x_2, x_3)$ *and* $\mathbf{y} = (y_1, y_2, y_3)$.

- Proof. Integrating by parts gives

$$\begin{aligned} G(\mathbf{x}, \mathbf{y}) &= \int_\Omega \delta(\mathbf{z} - \mathbf{x}) G(\mathbf{z}, \mathbf{y}) d\mathbf{z} = -\int_\Omega \nabla_{\mathbf{z}}^2 G(\mathbf{z}, \mathbf{x}) \; G(\mathbf{z}, \mathbf{y}) d\mathbf{z} \\ &= \int_\Omega \nabla_{\mathbf{z}} G(\mathbf{z}, \mathbf{x}) \cdot \nabla_{\mathbf{z}} G(\mathbf{z}, \mathbf{y}) d\mathbf{z} \\ &= -\int_\Omega G(\mathbf{z}, \mathbf{x}) \cdot \nabla_{\mathbf{z}}^2 G(\mathbf{z}, \mathbf{y}) d\mathbf{z} = \int_\Omega G(\mathbf{z}, \mathbf{x}) \; \delta(\mathbf{z} - \mathbf{y}) d\mathbf{z} \\ &= G(\mathbf{y}, \mathbf{x}). \end{aligned}$$

Theorem (Poisson's equation with Dirichlet BC)

For $f \in C^1(\partial\Omega)$, the solution of the Poisson's equation

$$\begin{cases} -\nabla^2 u = \rho & \text{in } \Omega \\ u|_{\partial\Omega} = f. \end{cases}$$

is expressed as

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' + \int_{\partial\Omega} K(\mathbf{r};\mathbf{r}')\, f(\mathbf{r}')\, dS_{\mathbf{r}'}, \quad \mathbf{r} \in \Omega$$

where the Poisson kernel $K(\mathbf{r},\mathbf{r}')$ is

$$K(\mathbf{r},\mathbf{r}') = -\mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}'), \qquad (\forall \mathbf{r}' \in \partial\Omega \;\&\; \forall \mathbf{r} \in \Omega).$$

- For a fixed $\mathbf{r} \in \Omega$, integration by parts gives

$$\begin{aligned} u(\mathbf{r}) &= \int_\Omega \delta(\mathbf{r}-\mathbf{r}')\, u(\mathbf{r}')d\mathbf{r}' = -\int_\Omega \nabla^2_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}')\, u(\mathbf{r}')d\mathbf{r}' \qquad (\text{ distributional sense}) \\ &= \int_{\partial\Omega} K(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\, dS_{\mathbf{r}'} + \int_\Omega \nabla_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}') \cdot \nabla u(\mathbf{r}')d\mathbf{r}' \\ &= \int_{\partial\Omega} K(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\, dS_{\mathbf{r}'} + \int_\Omega G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}'. \end{aligned}$$

Theorem (Green's function in a ball)

The Green's function of the ball $\Omega = \{\mathbf{r} = (x, y, z) \ : \ |\mathbf{r}| < s\}$ *is given by*

$$G(\mathbf{r};\mathbf{r}') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} - \underbrace{\frac{s}{4\pi|\mathbf{r}'|\left|\mathbf{r}-\frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}\right|}}_{H(\mathbf{r},\mathbf{r}')}.$$

- The image point $\mathbf{r}'_*$ with respect to the sphere $\partial\Omega$ is given by

$$\mathbf{r}'_* = \frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}.$$

As a result,

$$H(\mathbf{r},\mathbf{r}') = \frac{s}{4\pi|\mathbf{r}'|\left|\mathbf{r}-\frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}\right|}$$

is harmonic w.r.t. $\mathbf{r}$ and

$$\frac{s}{4\pi|\mathbf{r}'|\left|\mathbf{r}-\frac{s^2\mathbf{r}'}{|\mathbf{r}'|^2}\right|} = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad \text{for all } |\mathbf{r}| = s.$$

Theorem (Mean value property and maximum principle)

Assume that $u \in C^2(\bar{\Omega})$ satisfies $\nabla^2 u(\mathbf{r}) = 0$ in Ω. For each $\mathbf{r}^* \in \Omega$, $u(\mathbf{r}^*)$ is the average of u over its neighboring sphere centered at $\mathbf{r}^*$:

$$u(\mathbf{r}^*) = \frac{1}{4\pi s^2} \int_{\partial B_s(\mathbf{r}^*)} u(\mathbf{r}) dS_{\mathbf{r}} \qquad (\forall B_s(\mathbf{r}^*) \subset \Omega).$$

Moreover, u can not have a strict local maximum or minimum at any interior point of Ω. In other words,

$$\sup_{\partial\Omega} u = \sup_{\Omega} u \qquad \textit{and} \qquad \min_{\partial\Omega} u = \min_{\Omega} u.$$

- Assume $B_s(\mathbf{r}^*) \subset \Omega$. If $K(\mathbf{r}, \mathbf{r}')$ is the Poisson Kernel for the ball $B_s(\mathbf{r}^*)$, a straightforward computation gives

$$K(\mathbf{r}, \mathbf{r}^*)|_{\mathbf{r} \in B_s(\mathbf{r}^*)} = \text{ constant} = \frac{1}{4\pi s^2}.$$

Then, the mean value property follows from the representation formula

$$u(\mathbf{r}) = \int_{\Omega} G(\mathbf{r}; \mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' + \int_{\partial\Omega} K(\mathbf{r}; \mathbf{r}')\ f(\mathbf{r}')\ dS_{\mathbf{r}'}$$

with $\rho = 0$ and the above identity. Moreover, the mean value property gives the maximum principle.

Theorem

Denote $B_s = \{|\mathbf{r}| < s\}$. *If* $u \in C^2(\bar{B}_s)$ *satisfies* $-\nabla^2 u = \rho$ *in* B_s, *then*

$$u(0) = ave_{\partial B_s} u \; + \int_{B_s} \frac{1}{4\pi} \left(\frac{1}{|\mathbf{r}|} - \frac{1}{s} \right) \rho(\mathbf{r}) d\mathbf{r}$$

where $ave_{\partial B_s} u = \frac{1}{4\pi s^2} \int_{\partial B_s} u(\mathbf{r}) dS_{\mathbf{r}}$. *In particular,*

$$\nabla^2 u = \rho \geq 0 \;\; in\ B_s \Rightarrow \;\; u(0) \leq ave_{\partial B_s} u,$$
$$\nabla^2 u = \rho \leq 0 \;\; in\ B_s \Rightarrow \;\; u(0) \geq ave_{\partial B_s} u.$$

- The identity follows from the representation formula

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r};\mathbf{r}')\rho(\mathbf{r}')d\mathbf{r}' \; + \int_{\partial\Omega} K(\mathbf{r};\mathbf{r}')\, f(\mathbf{r}')\, dS_{\mathbf{r}'}$$