Laplace Equation in 3D
Green's Function

$$u_{xx} + u_{yy} + u_{zz} = 0$$

Theorem (Green's function for $\mathbb{R}^3$)

The function $\frac{1}{4\pi|\mathbf{r}|}$ *satisfies the following properties:*

$$
\begin{aligned}
-\nabla^2\left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) &= \delta(\mathbf{r}-\mathbf{r}'), \\
-\nabla^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' &= \rho(\mathbf{r}) \qquad (\forall\ \rho \in C_0^2(\mathbb{R}^3)).
\end{aligned}
$$

$$\nabla^2\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0$$

for $(x, y, z) \neq 0$

- Let $\rho \in C_0^2(\mathbb{R}^3)$. Straightforward computation gives

$$-\nabla^2\left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = 0 \qquad \text{if} \quad \mathbf{r} \neq \mathbf{r}'.$$

- Hence, for an arbitrary ball B containing $\mathbf{r}$, we have

$$\nabla^2 \int_{\mathbb{R}^3 \setminus B} \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' = 0 \qquad (\ \forall B \ni \mathbf{r}). \tag{1}$$

This leads to

$$-\nabla^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' = -\nabla^2 \int_{B} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' \qquad (\ B \ni \mathbf{r}).$$

Proof of $-\nabla^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' = \rho(\mathbf{r})$ for all $\rho \in C_0^2(\mathbb{R}^3)$

Recall $-\nabla^2\left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = 0$ for $\mathbf{r} \neq \mathbf{r}'$. For $\mathbf{r} \in B$,

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$$
\begin{aligned}
-\nabla^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' &= -\nabla^2 \int_B \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' \\
&= \lim_{\epsilon\to 0+} \int_B \nabla_{\mathbf{r}}^2 \left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)}\right)\rho(\mathbf{r}')d\mathbf{r}' \quad (\epsilon \text{ allows pushing } \nabla^2 \text{ inside } \textstyle\int_B) \\
&= \rho(\mathbf{r}) \lim_{\epsilon\to 0+} \int_B \nabla_{\mathbf{r}}^2 \left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)}\right) d\mathbf{r}' \\
&\qquad + \underbrace{\lim_{\epsilon\to 0+} \int_B \nabla_{\mathbf{r}}^2 \left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)}\right) \underbrace{(\rho(\mathbf{r}')-\rho(\mathbf{r}))}_{O(|\mathbf{r}-\mathbf{r}'|)} d\mathbf{r}'}_{O(|\mathbf{r}-\mathbf{r}'|) \text{ allows pushing } \lim_{\epsilon\to 0+} \text{ inside } \int_B \text{ \& this converges to zero.}}
\end{aligned}
$$

- Hence, it suffices to prove

$$
\lim_{\epsilon\to 0+} \int_B \nabla_{\mathbf{r}}^2 \left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)}\right) d\mathbf{r}' = -\nabla^2 \int_B \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = 1 \qquad (\forall B \ni \mathbf{r}).
$$

$$\nabla^2 \int_{\mathbb{R}^3 \setminus \boldsymbol{B}} \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' = 0 \qquad (\forall \boldsymbol{B} \ni \mathbf{r})$$

$$-\nabla^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' = -\nabla^2 \int_{B} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}'$$

$\boldsymbol{r} \in \boldsymbol{B}$

$$= \lim_{\epsilon \to 0+} \int_B \nabla^2_{\mathbf{r}} \left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)} \right) \rho(\mathbf{r}')d\mathbf{r}' \quad (\epsilon \text{ allows pushing } \nabla^2 \text{ inside } \textstyle\int_B)$$

Lebesgue's Dominated Convergence Theorem (a modified version for this lecture).

Let $\boldsymbol{\Omega}$ be a subdomain of R^n. If $\{f_n\} \subset L^1(\Omega)$ is a sequence satisfying

$$\sup_n |f_n(x)| \le g(x) \quad \& \quad \lim_{n\to\infty} f_n(x) = f(x) \qquad a.e.$$

with $f, g \in L^1(\Omega)$ (i.e. $\int_{R^3} |f| < \infty$ & $\int_{R^3} |g| < \infty$), then

$$\lim_{n\to\infty} \int_\Omega f_n = \int_\Omega \lim_{n\to\infty} f_n$$

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$$\nabla \int_\Omega \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \right) \rho\, dr = \int_\Omega \nabla \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \right) \rho\, dr$$

$$\nabla^2 \int_\Omega \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \right) \rho dr \ne \int_\Omega \nabla^2 \left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \right) \rho dr$$

In 3-D, $\int_B \frac{C}{|r|^1} < \infty$, $\int_B \frac{C}{|r|^2} < \infty$, $\int_B \frac{C}{|r|^3} = \infty$

$$= \lim_{\epsilon\to 0+}\int_B \nabla^2_{\mathbf{r}}\left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)}\right)\rho(\mathbf{r}')d\mathbf{r}' \quad (\epsilon \text{ allows pushing } \nabla^2 \text{ inside } \textstyle\int_B)$$

$$-\nabla^2\int_{\mathbb{R}^3}\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}')d\mathbf{r}' = \rho(\mathbf{r})\lim_{\epsilon\to 0+}\int_B \nabla^2_{\mathbf{r}}\left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)}\right)d\mathbf{r}'$$

$$\underbrace{\boxed{+\lim_{\epsilon\to 0+}\int_B \nabla^2_{\mathbf{r}}\left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)}\right)\underbrace{(\rho(\mathbf{r}')-\rho(\mathbf{r}))}_{O(|\mathbf{r}-\mathbf{r}'|)}d\mathbf{r}'}}_{O(|\mathbf{r}-\mathbf{r}'|) \text{ allows pushing } \lim_{\epsilon\to 0+} \text{ inside } \int_B \text{ \& this converges to zero.}}$$

0

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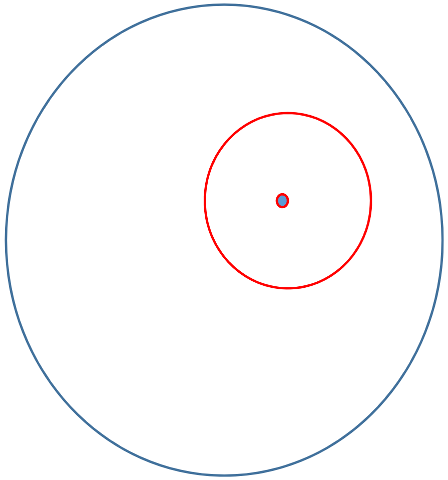

Proof of $-\nabla^2 \int_B \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = 1 \quad (\forall B \ni \mathbf{r})$.

Let $B_r(\mathbf{r})$ be a ball with the center $\mathbf{r}$ and the radius r such that $B_r(\mathbf{r}) \subset B$.

$$\nabla^2 \int_B \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \nabla \cdot \int_B \nabla_{\mathbf{r}} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) d\mathbf{r}'$$

$$= \underbrace{-\nabla \cdot \int_B \nabla_{\mathbf{r}'} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) d\mathbf{r}' = -\nabla \cdot \int_{\partial B} \mathbf{n}(\mathbf{r}') \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) dS_{\mathbf{r}'}}_{\text{diveregnce theorem}}$$

$$= -\int_{\partial B} \mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) dS_{\mathbf{r}'}$$

$$= \underbrace{\int_{\partial B} \mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) dS_{\mathbf{r}'} = \int_{\partial B_r(\mathbf{r})} \mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) dS_{\mathbf{r}'}}_{\int_{B\setminus B_r(\mathbf{r})} \nabla^2 \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}'=0 \quad \text{since } \nabla^2 \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} = 0 \text{ for } \mathbf{r} \neq \mathbf{r}'}$$

$$= \int_{\partial B_r(\mathbf{r})} \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|} \cdot \left(\frac{\mathbf{r}'-\mathbf{r}}{4\pi|\mathbf{r}-\mathbf{r}'|^3}\right) dS_{\mathbf{r}'} = 1$$

Hence, $-\nabla^2 \left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) = \delta(\mathbf{r}-\mathbf{r}')$.

$$\nabla^2 \int_B \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = \nabla \cdot \int_B \nabla_{\mathbf{r}} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} \right) d\mathbf{r}'$$

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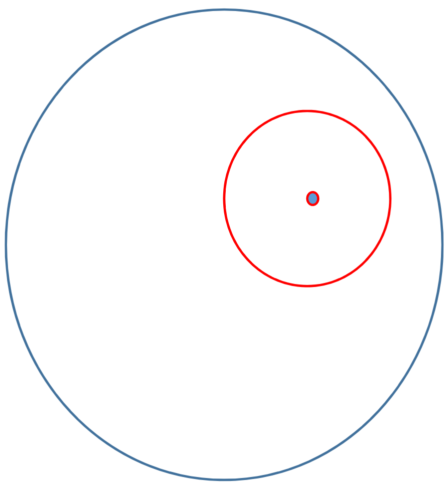
$$\nabla^2 \int_B \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = -\int_{\partial B} \mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} \right) dS_{\mathbf{r}'}$$

$$= \underbrace{\int_{\partial B} \mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} \right) dS_{\mathbf{r}'} = \int_{\partial B_r(\mathbf{r})} \mathbf{n}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \left(\frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} \right) dS_{\mathbf{r}'}}_{\int_{B\setminus B_r(\mathbf{r})} \nabla^2 \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' = 0}$$

since $\nabla^2 \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|} = 0$ for $\mathbf{r} \neq \mathbf{r}'$

$$= \int_{\partial B_r(\mathbf{r})} \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|} \cdot \left(\frac{\mathbf{r}'-\mathbf{r}}{4\pi|\mathbf{r}-\mathbf{r}'|^3} \right) dS_{\mathbf{r}'} = 1$$



Theorem (Green's function for $\mathbb{R}^3$)

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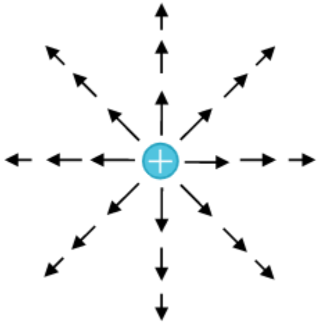
$$-\nabla^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \rho(\mathbf{r}') d\mathbf{r}' = \rho(\mathbf{r})$$

$$\boxed{-\nabla^2 \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \rho(\mathbf{r}') d\mathbf{r}' = \rho(\mathbf{r}) \lim_{\epsilon \to 0+} \int_B \nabla_{\mathbf{r}}^2 \left(\frac{-1}{4\pi(|\mathbf{r}-\mathbf{r}'|+\epsilon)} \right) d\mathbf{r}'}$$



Electrostatics

- The electric field intensity from a point charge q, located at $\mathbf{r}_0$ in an unbounded free space can be expressed as

$$\mathbf{E}(\mathbf{r}) = \frac{q}{\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_0}{4\pi|\mathbf{r} - \mathbf{r}_0|^3} \qquad \text{(based on Coulomb's law)}$$

where $\epsilon_0 = 8.85 \times 10^{-12}$ is the permittivity of the free space.

- In general, we can express **E** subject to a certain charge distribution $\rho(x)$ in Ω as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{\epsilon_0} \int_\Omega \frac{\mathbf{r} - \mathbf{r}'}{4\pi|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}') d\mathbf{r}'.$$

- Hence, **E** can be expressed as

$$\mathbf{E}(\mathbf{r}) = -\nabla u(\mathbf{r}) \quad \text{where} \quad u(\mathbf{r}) = \int_{\mathbb{R}^3} \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}')\, d\mathbf{r}'.$$

- Here, u is the electric potential and it satisfies the Poisson's equation:

$$-\nabla^2 u(\mathbf{r}) = \rho(\mathbf{r}).$$