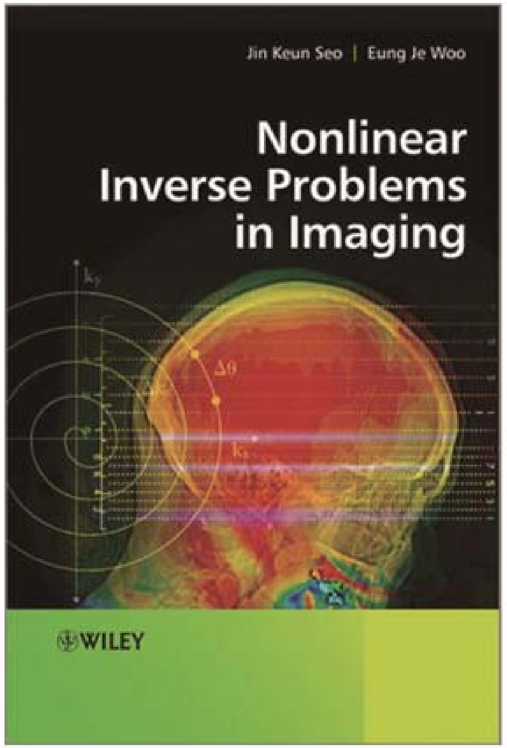
[Lecture 7]

Variational formulation of elliptic PDEs

Lax-Milgram/Riesz Representation



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Slide & video
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Example: Poisson's equation

Define the map $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a(u, \phi) = \int_\Omega \sigma \nabla u \cdot \nabla \phi \qquad (0 < c_0 < \sigma(\mathbf{r}) < c_1 < \infty)$$

and define $b : H_0^1(\Omega) \to \mathbb{R}$ by

$$b(\phi) = \int_\Omega f \phi dx.$$

The solvability problem of $\begin{cases} -\nabla \cdot (\sigma \nabla u) = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$ is equivalent to the uniqueness and existence question of finding $u \in X = H_0^1(\Omega)$ satisfying

$$a(u, \phi) = b(\phi), \quad \forall \phi \in X.$$

Note that the map $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ satisfies the followings:

- both $a(u, \cdot) : X \to \mathbb{R}$ and $a(\cdot, w) : X \to \mathbb{R}$ are linear for all $u, w \in X$,
- $|a(u, v)| < c_1 \|u\| \|v\|$ and $c_0 \|u\|^2 \leq a(u, u)$.

Solvability of $\begin{cases} -\nabla\cdot(\sigma\nabla u) = f & \text{in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$

Example: Finite Element Method (FEM) for solving Poisson's Equation

Assuming that the Hilbert space $H_0^1(\Omega)$ has a basis $\{\phi_k \ : \ k = 1, 2, \cdots\}$, the variational problem $[a(u,\phi) = b(\phi), \forall\phi \in X]$ is to determine the coefficient $\{u_k : k = 1, 2, \cdots\}$ of $u = \sum u_k\phi_k$ satisfying

$$\underbrace{a(\sum_k u_k\phi_k, \phi_j)}_{\int_\Omega \sigma\nabla u\cdot\nabla\phi_j dx} = \underbrace{b(\phi_j)}_{\int_\Omega f\phi_j dx}, \quad \forall j = 1, 2, \cdots .$$

$$a(u,\phi) = \int_{\Omega} \sigma \nabla u \cdot \nabla \phi \, dx$$

$$b(\phi) = \int_{\Omega} f \phi \, dx$$

$$\boxed{a(u, \phi_k) = b(\phi_k)}$$

for $k = 1, 2, \cdots$

$$u = \sum_i u_i \phi_i$$

$$\boxed{a\left(\sum_i u_i \phi_i, \phi_k\right) = b(\phi_k)}$$

$$\boxed{\sum_i u_i \, a(\phi_i, \phi_k) = b(\phi_k)}$$

$$\begin{bmatrix} a(\phi_1,\phi_k), a(\phi_2,\phi_k) \cdots a(\phi_j,\phi_k) \cdots \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ \vdots \end{bmatrix} = b(\phi_k)$$

$$a(u, \phi_k) = b(\phi_k) \quad \text{for } k = 1, 2, \cdots$$

$$u = \sum_i u_i \phi_i$$

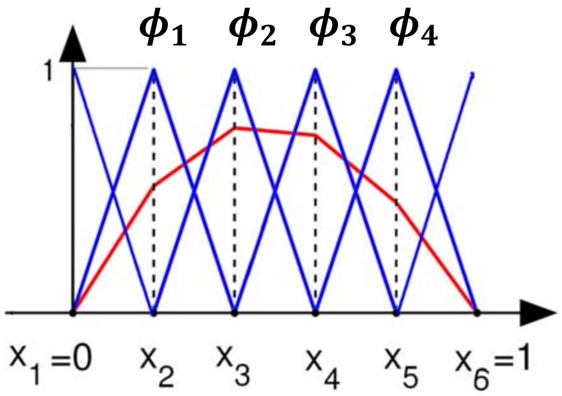
$$\underbrace{\begin{pmatrix} a(\phi_1, \phi_1) & a(\phi_1, \phi_2) & \cdots & \\ a(\phi_2, \phi_1) & a(\phi_2, \phi_2) & \cdots & \\ a(\phi_3, \phi_1) & a(\phi_3, \phi_2) & \cdots & \\ & & & \ddots \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix}}_{\mathbf{u}} = \underbrace{\begin{pmatrix} b(\phi_1) \\ b(\phi_2) \\ b(\phi_3) \\ \vdots \end{pmatrix}}_{\mathbf{b}}$$

One-dimensional FEM for solving Poisson equation

(abuse notation: $\nabla = \frac{d}{dx}$)

$$-\nabla \cdot \nabla u = 1 \quad in\ (0,1)$$
$$u(0) = 0, u(1) = 0$$

$$\underbrace{\int_0^1 u' \phi_j'\, dx}_{a(u,\phi_j)} = \underbrace{\int_0^1 \phi_j dx}_{b(\phi_j)} \quad for\ j = 1,2,3,4$$



coefficients

Assume $u(x) \approx u_1\, \phi_1(x) + u_2\, \phi_2(x) + u_3\, \phi_3(x) + u_4\, \phi_4(x)$

$$\begin{pmatrix} a(\phi_1,\phi_1) & a(\phi_2,\phi_1) & a(\phi_3,\phi_1) & a(\phi_4,\phi_1) \\ a(\phi_1,\phi_2) & a(\phi_2,\phi_2) & a(\phi_3,\phi_2) & a(\phi_4,\phi_2) \\ a(\phi_1,\phi_3) & a(\phi_2,\phi_3) & a(\phi_3,\phi_3) & a(\phi_4,\phi_3) \\ a(\phi_1,\phi_4) & a(\phi_2,4) & a(\phi_3,\phi_4) & a(\phi_4,\phi_4) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_2 \\ u_4 \end{pmatrix} = \begin{pmatrix} b(\phi_1) \\ b(\phi_2) \\ b(\phi_3) \\ b(\phi_4) \end{pmatrix}$$

$$a(\phi_j,\phi_k) = \int_0^1 \nabla\phi_j(x) \cdot \nabla\phi_k(x)\, dx = \begin{cases} 10 & if\ \ j = k \\ -5 & if\ |j-k| = 1\,, \\ 0 & otherwise \end{cases}$$

$$b(\phi_j) = \int_0^1 \phi_j(x) dx = \frac{1}{5}$$

$\boldsymbol{X = \mathbf{R}^n}$

Hibert space

$\boldsymbol{a(u,\phi) = \langle Au, \phi \rangle}$

$A: n \times n$ positive definite matrix.

$\boldsymbol{a}(\mathrm{u}, \mathrm{u}) \geq C\, ||u||^2 \quad \forall u \in X$

$\boldsymbol{b(\phi) = f \cdot \phi}$

$\boldsymbol{a}: \mathrm{X} \rightarrow R$ is linear.

Today, we will show the solvability by proving the following relations:

$$\boxed{Au = f} \quad \xleftrightarrow{\text{definition}} \quad \boxed{a(u,\phi) = b(\phi) \quad \forall \phi \in X}$$

Lax-Milgram Theorem

$$\boxed{u = arg \min_{u \in X} \Phi(u) := \frac{1}{2} a(u,u) - b(u)}$$

$$\boldsymbol{\Phi(u) = \frac{1}{2} \langle Au, u \rangle - f \cdot u}$$

$$\Downarrow$$

$$\boldsymbol{\nabla \Phi(u) = \mathrm{Au - f}}$$

$\boldsymbol{X = \mathrm{H}_0^1(\Omega)}$

Hibert space

$$\boldsymbol{a(u,\phi) = \int_{\Omega} \sigma \nabla u \cdot \nabla \phi \, dx}$$

$\boldsymbol{a}: \mathrm{X} \times X \rightarrow R$ is bilinear.

$\boldsymbol{a}(\mathrm{u},\mathrm{u}) \geq C \, || u ||^2 \;\; \forall u \in X$

Poincare inequality

$$\boldsymbol{b(\phi) = \int_{\Omega} f \phi \, dx}$$

$\boldsymbol{a}: \mathrm{X} \rightarrow R$ is linear.

Today, we will show the solvability by proving the following relations:

$$\boxed{\begin{gathered} \nabla \cdot (\sigma \nabla u) = f \;\; in \;\; \Omega \\ u|_{\partial\Omega} = 0 \end{gathered}}$$

definition

$$\boxed{a(u,\phi) = b(\phi) \quad \forall \phi \in X}$$

Lax-Milgram Theorem

$$\boxed{u = arg \min_{u \in X} \Phi(u) := \frac{1}{2} a(u,u) - b(u)}$$

Theorem (Lax-Milgram theorem)

Suppose the bounded bilinear map $a : X \times X \to \mathbb{R}$ is symmetric and

$$a(u,u) \geq c\|u\|^2, \quad c > 0.$$

Suppose $b(\cdot) : X \to \mathbb{R}$ is linear.

- *There exists a unique solution $u \in X$ of the minimization problem*

$$\textit{Minimize} \quad \Phi(u) := \frac{1}{2} a(u,u) - b(u), \quad u \in X.$$

- *The solution u of the minimization problem is the solution of variational problem:*

$$a(u,\phi) = b(\phi), \quad \forall \phi \in X.$$

Example: Consider the solution $u \in X = H_0^1(\Omega)$ of Poisson's equation $-\nabla^2 u = f$ in Ω. This u is a minimizer of the problem:

$$\min \quad \Phi(u) := \frac{1}{2} \underbrace{\int_\Omega |\nabla u(x)|^2 dx}_{a(u,u)} - \underbrace{\int_\Omega f(x)u(x)dx}_{b(u)}, \qquad u \in X = H_0^1(\Omega).$$

Proof of Lax-Milgram: Existence

Existence of the minimizer can be proven by showing

$$\exists\, u \in X \quad s.t.\;\; \Phi(u) = \alpha := \inf_{v \in X} \Phi(v).$$

Proof. Let $\{u_n\}$ be a sequence s.t. $\Phi(u_n) \to \alpha$.

- $a(u_n, u_n) + a(u_m, u_m) = \frac{1}{2} a(u_n - u_m, u_n - u_m) + \frac{1}{2} a(u_n + u_m, u_n + u_m)$.
- $2\Phi(u_n) + 2\Phi(u_m) = \frac{1}{2} a(u_n - u_m, u_n - u_m) + 4\Phi(\frac{u_n + u_m}{2})$.
- Setting $n, m \to \infty$, we have

$$\begin{aligned} 4\alpha \longleftarrow 2\Phi(u_n) + 2\Phi(u_m) &= \frac{1}{2} a(u_n - u_m, u_n - u_m) + 4\Phi(\frac{u_n + u_m}{2}) \\ &\geq \frac{c}{2} \|u_n - u_m\|^2 + 4\alpha. \end{aligned}$$

- Hence $\{u_n\}$ is a Cauchy sequence and $\exists u \in X$ so that $u = \lim_{n\to\infty} u_n$. Since $\Phi(\cdot)$ is continuous due to the assumption that $a(\cdot,\cdot)$ and $b(\cdot)$ are bounded and linear, we prove existence of the minimizer u:

$$\Phi(u) = \lim_{n\to\infty} \Phi(u_n) = \alpha.$$

$$\Phi(u) = \frac{1}{2} a(u,u) - b(u)$$

$$a(u_n, u_n) + a(u_m, u_m) = \underbrace{\frac{1}{2} a(u_n - u_m, u_n - u_m)}_{\frac{1}{2}a(u_n,u_n) - \cancel{a(u_n,u_m)} + \frac{1}{2}a(u_m,u_m)} + \underbrace{\frac{1}{2} a(u_n + u_m, u_n + u_m)}_{\frac{1}{2}a(u_n,u_n) + \cancel{a(u_n,u_m)} + \frac{1}{2}a(u_m,u_m)}.$$

$$2\Phi(u_n) + 2\Phi(u_m) = \underbrace{\frac{1}{2} a(u_n - u_m, u_n - u_m)}_{\underbrace{a(u_n,u_n)}_{2\Phi(u_n) + \cancel{2b(u_n)}} + \underbrace{a(u_m,u_m)}_{2\Phi(u_m) + \cancel{2b(u_m)}} - \underbrace{\cancel{\tfrac{1}{2}}^{2} a\left(\frac{u_n+u_m}{2}, \frac{u_n+u_m}{2}\right)}_{4\Phi\left(\frac{u_n+u_m}{2}\right) - \cancel{2b(u_n+u_m)}}} + 4\Phi\left(\frac{u_n + u_m}{2}\right).$$

Proof of Lax-Milgram: Uniqueness

- If u is a minimizer,

$$\frac{d}{dt}\Phi(u+t\phi)|_{t=0} = 0 \quad (\forall \phi \in X) \qquad \Longrightarrow \qquad a(u,\phi) = b(\phi) \quad (\forall \phi \in X).$$

- Hence, if u and v are minimizers, then

$$a(u,\phi) = b(\phi) = a(v,\phi) \qquad (\forall \phi \in X)$$

  which is equivalent to

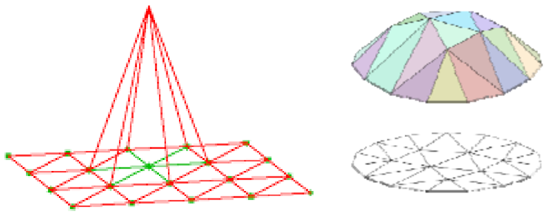
$$a(u-v,\phi) = 0 \qquad (\forall \phi \in X).$$

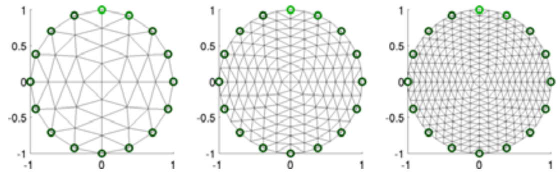
- In particular, $a(u-v,u-v) = 0$. Then, it follows from **the condition of** $a(\cdot,\cdot)$ that

$$c\|u-v\|^2 \le a(u-v,u-v) = 0,$$

  which gives $u = v$.

Ritz Approach





Taking account of the finite element method, we assume the followings:

- X is a real Hilbert space with a norm $\|\cdot\|$.
- $\{X_n\}$ is a sequence of a finite dimensional subspace of X such that

$$X_n \subset X_{n+1} \qquad \& \qquad \overline{\cup_{n=1}^{\infty} X_n} = X.$$

- $\{\phi_j^n : j = 1, \cdots, N_n\}$ is a basis of X_n.
- $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be a bounded, symmetric, strongly positive, bilinear map and

$$|a(u,v)| \le M\|u\|\|v\|, \quad c\|u\|^2 \le a(u,u), \quad \forall u, v \in X.$$

- $b \in X^*$ where X^* is the set of linear functional on X.

Consider the minimization problem:

$$\min \quad \Phi(u_n) := \frac{1}{2} a(u_n, u_n) - b(u_n), \quad u_n \in X_n.$$

The variational problem is as follows:

$$\text{Find} \quad u_n \in X_n \quad s.t. \quad a(u_n, \phi_n) = b(\phi_n), \quad \forall \phi_n \in X_n.$$

Example: Consider the solution $u \in X = H_0^1(\Omega)$ of Poisson's equation $-\nabla^2 u = f$ in Ω.
This u is a minimizer of the problem:

$$\min \quad \Phi(u) := \frac{1}{2} \underbrace{\int_\Omega |\nabla u(x)|^2 dx}_{a(u,u)} - \underbrace{\int_\Omega f(x)u(x)dx}_{b(u)}, \qquad u \in X = H_0^1(\Omega).$$

Theorem (Projection theorem)

The space X equipped with the new inner product and norm

$$\langle u, v\rangle_a := a(u, v), \quad \|u\|_a := \sqrt{a(u, u)} \qquad (\forall \ u, v \in X)$$

is also a Hilbert space. For any closed subspace $V = \overline{V} \subset X$ and for each $u \in X$,

$$\exists \ u_V \in \quad s.t. \quad \|u - u_V\|_a = \min_{v \in V} \|u - v\|_a.$$

Denoting $V^{\perp_a} := \{w \in X \ : \ a(w, v) = 0. \ \forall \ v \in V\}$, if u is decomposed as

$$u = u_1 + u_2, \quad (u_1 \in V, \ \ u_2 \in V^{\perp_a}),$$

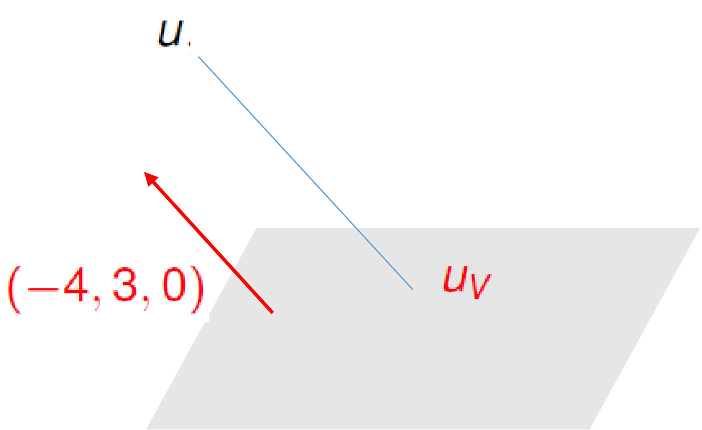
then $u_1 = u_V$ and $u_2 = u - u_V$.

Example: Let $X = \mathbb{R}^3$ with its new inner product given by

$$\langle u, v\rangle_a := \langle \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} u \ , \ v \ \rangle \qquad (\forall \ u, v \in X = \mathbb{R}^3)$$

Let $V = \text{span}\{(1, 1, 0), (0, 0, 1)\} = \{v \in X \ : \ (-1, 1, 0) \cdot v = 0\}$. Then, $V^{\perp_a} := \text{span}\{(-4, 3, 0)\}$. For any $u \in X$, $u_{V^{\perp_a}} = \alpha(-4, 3, 0)$ for some α. This is the key of Riesz Representation:

$$\exists \text{ unique } u \in X \ \ s.t. \ \langle u, v\rangle_a = \underbrace{(-1, 1, 0) \cdot v}_{\text{linear functional}} \qquad (\forall v \in X)$$



$V = \text{span}\{(1, 1, 0), (0, 0, 1)\} = \{v \in X \ : \ (-1, 1, 0) \cdot v = 0\}$

Proof of $\exists\, u_V \in V \quad \text{s.t.} \quad \|u - u_V\|_a = \min_{v \in V} \|u - v\|_a$

- Let u be fixed. Since

$$\|u - v\|_a^2 = a(v, v) - 2a(u, v) + a(u, u),$$

the problem is equivalent to solve the following minimization problem:

$$\min_{v \in V} \quad \tilde{\Phi}(v) := \frac{1}{2}a(v, v) - b(v) \qquad b(v) = a(u, v)$$

- According to the Lax-Milgram theorem, $\exists u_V \in V$ of the minimization problem.

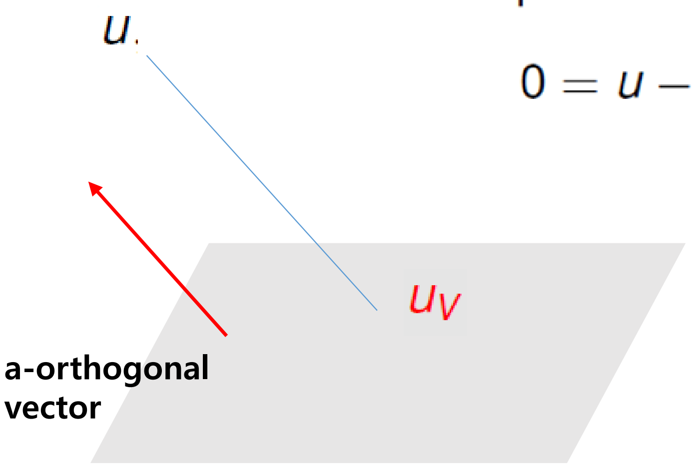
Proof of $\exists\, u_V \in V$ s.t. $\|u - u_V\|_a = \min_{v \in V} \|u - v\|_a$

- According to the Lax-Milgram theorem, $\exists u_V \in V$ of the minimization problem.
- Next, we will show $u - u_V \in V^{\perp}$.

$$
\begin{aligned}
& \|u - u_V\|_a^2 \le \|u - (u_V + tv)\|_a^2 \quad (\forall\, v \in V,\ \ \forall\, t \in \mathbb{R}) \\
\Rightarrow\quad & 0 \le -2ta(u - u_V, v) + t^2 a(v, v) \quad (\forall\, v \in V,\ \ \forall\, t \in \mathbb{R}) \\
\Rightarrow\quad & -t\, a(v, v) \le 2a(u - u_V, v) \le t\, a(v, v), \quad (\forall\, v \in V,\ \ \forall\, t > 0).
\end{aligned}
$$

- Letting $t \to 0$, we get $a(u - u_V, v) = 0 \quad (\forall\, v \in V) \quad \Rightarrow \quad u - u_V \in V^{\perp_a}$.
- Uniqueness of the decomposition follows from the fact that

$$
0 = u - u = \underbrace{(u_1 - u_V)}_{\in\ V} + \underbrace{(u_2 - u + u_V)}_{\in\ V^{\perp_a}} \quad \Rightarrow \quad u_1 = u_V \ \ \&\ \ u_2 = u - u_V.
$$



Theorem (Riesz representation theorem)

For each linear functional $b \in X^*$, there is a unique $u_* \in X$ such that

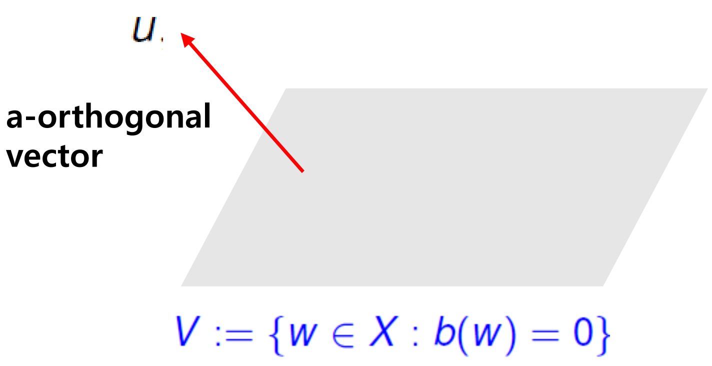
$$b(v) = a(u_*, v), \quad (\forall v \in X).$$

Moreover, $\|b\| = \sqrt{a(u_*, u_*)}$.

Assume $b \neq 0$. Let $V := \{w \in X : b(w) = 0\}$. Claim: $\dim V^{\perp_a} = 1$.

- Since $V^{\perp_a} \neq \emptyset$, $\exists u \in V^{\perp_a}$ s.t. $b(u) = 1$.
- Since $b(w - b(w)u) = b(w) - b(w)b(u) = 0$ for all $w \in X$,

$$w = \underbrace{w - b(w)u}_{\in \ V} + \underbrace{b(w)u}_{\in \ V^{\perp_a}} \qquad (\forall w \in X).$$



For each linear functional $b \in X^*$, there is a unique $u_* \in X$ such that

$$b(v) = a(u_*, v), \quad (\forall v \in X).$$

$$w = \underbrace{w - b(w)u}_{\in\ V} + \underbrace{b(w)u}_{\in\ V^{\perp_a}} \qquad (\forall w \in X).$$

Projection theorem $\Rightarrow$ the decomposition is unique & $\dim V^{\perp_a} = 1$.

For all $w \in X$, we have

$$a(u, w) = a(u, \underbrace{w - b(w)u}_{\in V} + b(w)u) = a(u, b(w)u) = a(u, u)b(w).$$

Hence, $u_* = u/a(u, u)$ is the solution!
Since $|b(w)| \le \|u_*\|_a \|w\|_a$ and $|b(u_*)| = \|u_*\|_a^2$, we have $\|b\| = \|u_*\|_a$.

Theorem (Ritz Error Estimate)

Assume that $u_n \in X_n$ is a unique solution of the minimization problem

$$\min \quad \Phi(u_n) := \frac{1}{2}a(u_n, u_n) - b(u_n), \quad u_n \in X_n.$$

Let u be a solution of the minimization problem with X_n replaced by X. Then,

$$(i) \quad \lim_{n\to\infty} \|u_n - u\| = 0,$$

$$(ii) \quad \|u - u_n\| \le \frac{M}{c} \min_{v\in X_n} \|u - v\|,$$

$$(iii) \quad \frac{c}{2}\|u - u_n\|^2 \le \Phi(u_n) - \Phi(u).$$

- $a(u, v) = b(v) = a(u_n, v)$ for all $v \in X_n$ implies $a(u - u_n, v) = 0 \quad (\forall v \in X_n)$.
- For all $v \in X_n$, we have

$$\|u - u_n\|^2 \le \frac{1}{c} a(u - u_n, u - u_n) = \frac{1}{c} a(u - u_n, u - v) \le \frac{M}{c} \|u - u_n\| \, \|u - v\|$$

  which gives (ii). (i) follows from the assumption that $X_n \to X$ as $n \to \infty$.
- For all $v \in X$,

$$\Phi(u + v) = \frac{1}{2} a(u + v, u + v) - b(u + v) = \frac{1}{2} a(v, v) + \underbrace{(a(u, v) - b(v))}_{=0} + \Phi(u)$$