Explicit & semi-implicit projection method Kim & Moin(1985) JCP
for time-dependent incompressible Navier-Stokes equations

Explicit & semi-implicit projection method for time-dependent incompressible Navier-Stokes equations

Kim & Moin(1985) JCP

Non-dimensionalized NS

$$\frac{\partial \vec{\boldsymbol{u}}}{\partial \mathrm{t}} = -\vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}} - \nabla \mathrm{p} + \frac{1}{Re} \nabla^2 \vec{\boldsymbol{u}}$$

$\nabla \cdot \vec{\boldsymbol{u}} = \mathbf{0}$ (Continuity equation)

$\vec{\boldsymbol{u}}$: **velocity**

p: **pressure**

Re : **Reynolds number**

Explicit & semi-implicit projection method for time-dependent incompressible Navier-Stokes equations

Kim & Moin(1985) JCP

Non-dimensionalized NS

$$\frac{\partial \vec{\boldsymbol{u}}}{\partial \mathrm{t}} = -\vec{\boldsymbol{u}} \cdot \nabla \vec{\boldsymbol{u}} - \nabla \mathrm{p} + \frac{1}{Re} \nabla^2 \vec{\boldsymbol{u}}$$

$\nabla \cdot \vec{\boldsymbol{u}} = \boldsymbol{0}$ (Continuity equation)

$\vec{\boldsymbol{u}}$: **velocity**

p: **pressure**

Re : **Reynolds number**

For Simplicity

$$\frac{\partial \vec{\boldsymbol{u}}}{\partial \mathrm{t}} = N(\vec{\boldsymbol{u}}) - G(\mathrm{p}) + \frac{1}{Re} L(\vec{\boldsymbol{u}})$$

$D(\vec{\boldsymbol{u}}) = \boldsymbol{0}$ (Continuity equation)

Explicit & semi-implicit projection method
for time-dependent incompressible Navier-Stokes equations

$$\frac{\partial \vec{\boldsymbol{u}}}{\partial \mathrm{t}} = N(\vec{\boldsymbol{u}}) - G(\mathrm{p}) + \frac{1}{Re} L(\vec{\boldsymbol{u}})$$

$D(\vec{\boldsymbol{u}}) = \mathbf{0}$ (Continuity equation)

Difficulties for solving the problem numerically?

- **Nonlinearity of the first equation**
- **No $\frac{\partial \boldsymbol{p}}{\partial \mathrm{t}}$ part, $\boldsymbol{p}$ as a Lagrange multiplier to make sure continuity eq.**

Explicit & semi-implicit projection method for time-dependent incompressible Navier-Stokes equations

Kim & Moin(1985) JCP

$$\frac{\partial \vec{\boldsymbol{u}}}{\partial \mathrm{t}} = N(\vec{\boldsymbol{u}}) - G(\mathrm{p}) + \frac{1}{Re} L(\vec{\boldsymbol{u}})$$

$D(\vec{\boldsymbol{u}}) = \mathbf{0}$ (Continuity equation)

Nonlinearity & Velocity-pressure coupled

Solve the coupled problem directly?

- **Difficult to be convergent**
- **Time consuming**

Decouple pressure and velocity?

Explicit & semi-implicit projection method for time-dependent incompressible Navier-Stokes equations

Kim & Moin(1985) JCP

$$\frac{\partial \vec{u}}{\partial t} = N(\vec{u}) - G(p) + \frac{1}{Re} L(\vec{u})$$

$$D(\vec{u}) = \mathbf{0} \text{ (Continuity equation)}$$

Nonlinearity & Velocity-pressure coupled

Solve the coupled problem directly?

- **Difficult to be convergent**
- **time consuming**

Decouple pressure and velocity?

[Hodge Decomposition]

Let Ω be a smooth, bounded domain, and $\hat{\mathbf{u}}$ be smooth vector field on Ω. The vector field $\hat{\mathbf{u}}$ can be decomposed in the form

$$\hat{\mathbf{u}} = \mathbf{u} + \nabla\phi$$

where

$$\nabla\cdot\mathbf{u} = 0, \qquad \mathbf{u}\cdot\mathbf{n} = 0 \ on\ \partial\Omega.$$

$$\Downarrow$$

Projection methods are proposed to solve time-dependent N-S equations.

Explicit & semi-implicit projection method for time-dependent incompressible Navier-Stokes equations

Kim & Moin(1985) JCP

$$\frac{\partial \vec{u}}{\partial t} = N(\vec{u}) - G(p) + \frac{1}{Re} L(\vec{u})$$

$D(\vec{u}) = \mathbf{0}$ (Continuity equation)

Nonlinearity & Velocity-pressure coupled

Solve the coupled problem directly?

- **Difficult to be convergent**
- **time consuming**

Decouple pressure and velocity?

Procedure of the projection method（3D）

- **Discretization form**

convection term:

2nd order adams-bashforth

Diffusion term:

Crank-Nicolson

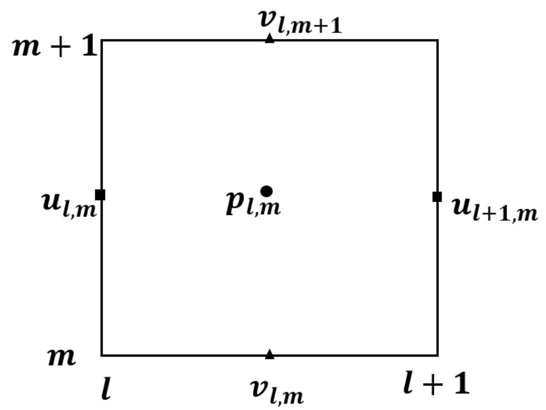
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left(3H_i^n - H_i^{n-1}\right) - \frac{\delta p^{n+1}}{\delta x_i} + \frac{1}{2Re} L\left(u_i^{n+1} + u_i^n\right)$$

$$\frac{\delta u_i^{n+1}}{\delta x_i} = 0$$

$$H_i^n = u_j^n \frac{\delta u_i^n}{\delta x_j} = u_1^n \frac{\delta u_i^n}{\delta x_1} + u_2^n \frac{\delta u_i^n}{\delta x_2} + u_3^n \frac{\delta u_i^n}{\delta x_3}$$

$$L\left(u_i^{n+1} + u_i^n\right) = \frac{\delta_j^2 (u_i^{n+1} + u_i^n)}{\delta x_j^2} = \frac{\delta_1^2 (u_i^{n+1} + u_i^n)}{\delta x_1^2} + \frac{\delta_2^2 (u_i^{n+1} + u_i^n)}{\delta x_2^2} + \frac{\delta_3^2 (u_i^{n+1} + u_i^n)}{\delta x_3^2}.$$

- **Apply Hodge decomposition**$\left(u_i^{*,n+1} = u_i^{n+1} + \Delta t \frac{\delta \varphi^{n+1}}{\delta x_i}\right)$



$$\frac{u_i^{*,n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left(3H_i^n - H_i^{n-1}\right) + \frac{1}{2Re} \frac{\delta_j^2 (u_i^{*,n+1} + u_i^n)}{\delta x_j^2}$$

$$\frac{u_i^{n+1} - u_i^{*,n+1}}{\Delta t} = -\frac{\delta \varphi^{n+1}}{\delta x_i}$$

$$\frac{\delta u_i^{n+1}}{\delta x_i} = 0$$

❑ **Calculate the momentum equation** $\frac{u_i^{*,n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left(3H_i^n - H_i^{n-1}\right) + \frac{1}{2Re}\frac{\delta_j^2(u_i^{*,n+1} + u_i^n)}{\delta x_j^2}$

$$\delta u_i^{*,n+1} = u_i^{*,n+1} - u_i^n$$

$$\delta u_i^{*,n+1} = -\frac{\Delta t}{2}\left(3H_i^n - H_i^{n-1}\right) + \frac{\Delta t}{2Re}\frac{\delta_j^2(\delta u_i^{*,n+1} + 2u_i^n)}{\delta x_j^2}$$

$$\left(1 - \frac{\Delta t}{2Re}\frac{\delta^2}{\delta x_j^2}\right)\delta u_i^{*,n+1} = -\frac{\Delta t}{2}\left(3H_i^n - H_i^{n-1}\right) + \frac{\Delta t}{Re}\frac{\delta_j^2 u_i^n}{\delta x_j^2}$$

$$A_j = \frac{\Delta t}{2Re}\frac{\delta_j^2}{\delta x_j^2}$$

$$(1 - A_1 - A_2 - A_3)\delta u_i^{*,n+1} = -\frac{\Delta t}{2}\left(3H_i^n - H_i^{n-1}\right) + 2(A_1 + A_2 + A_3)u_i^n$$

Apply velocity-velocity decoupling by $1 - A_1 - A_2 - A_3 = (1 - A_1)(1 - A_2)(1 - A_3) + O(\Delta t^2)$

$$(1 - A_1)(1 - A_2)(1 - A_3)\delta u_i^{*,n+1} = -\frac{\Delta t}{2}\left(3H_i^n - H_i^{n-1}\right) + 2(A_1 + A_2 + A_3)u_i^n$$

❑ **Solve** $\frac{u_i^{n+1} - u_i^{*,n+1}}{\Delta t} = -\frac{\delta \varphi^{n+1}}{\delta x_i}$ **for** φ^{n+1}

$$D\left(\frac{u_i^{n+1} - u_i^{*,n+1}}{\Delta t}\right) = D\left(-\frac{\delta \varphi^{n+1}}{\delta x_i}\right), \qquad D(\vec{\boldsymbol{u}}) = \boldsymbol{0}$$

$$D(u_i^{*,n+1}) = \Delta t\, D\left(-\frac{\delta \varphi^{n+1}}{\delta x_i}\right) \Longrightarrow D(u_i^{*,n+1}) = \Delta t \frac{\delta_i^2 \varphi^{n+1}}{\delta x_i^2}$$ Poisson equation

❑ **Solve** u_i^{n+1} **from** $\frac{u_i^{n+1} - u_i^{*,n+1}}{\Delta t} = -\frac{\delta \varphi^{n+1}}{\delta x_i}$

$$u_i^{n+1} = u_i^{*,n+1} - \Delta t \frac{\delta \varphi^{n+1}}{\delta x_i}$$

❑ **Solve** p^{n+1} **from** $p^{n+1} = \varphi^{n+1} - \frac{\Delta t}{2Re}\frac{\delta_i^2 \varphi^{n+1}}{\delta x_i^2}$

- $p^{n+1} = \varphi^{n+1} - \frac{\Delta t}{2Re}\frac{\delta_i^2 \varphi^{n+1}}{\delta x_i^2}$ **obtained from**

Original discretization

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left(3H_i^n - H_i^{n-1}\right) - \frac{\delta p^{n+1}}{\delta x_i} + \frac{1}{2Re}\frac{\delta_j^2 (u_i^{n+1} + u_i^n)}{\delta x_j^2}}$$

Present discretization

$$\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\frac{1}{2}\left(3H_i^n - H_i^{n-1}\right) - \frac{\delta \varphi^{n+1}}{\delta x_i} + \frac{1}{2Re}\frac{\delta_j^2 (u_i^{*,n+1} + u_i^n)}{\delta x_j^2}}$$

$$\frac{u_i^{n+1} - u_i^{*,n+1}}{\Delta t} = -\frac{\delta \varphi^{n+1}}{\delta x_i}$$

$$\boxed{\begin{aligned} \frac{\delta p^{n+1}}{\delta x_i} &= \frac{\delta \varphi^{n+1}}{\delta x_i} + \frac{1}{2Re}\frac{\delta_j^2 (u_i^{n+1} - u_i^{*,n+1})}{\delta x_j^2} \\ &= \frac{\delta \varphi^{n+1}}{\delta x_i} + \frac{\Delta t}{2Re}\frac{\delta_j^2 \left(-\frac{\delta \varphi^{n+1}}{\delta x_i}\right)}{\delta x_j^2} \\ &= \frac{\delta}{\delta x_i}\left(\varphi^{n+1} - \frac{\Delta t}{2Re}\frac{\delta_i^2 \varphi^{n+1}}{\delta x_i^2}\right) \end{aligned}}$$