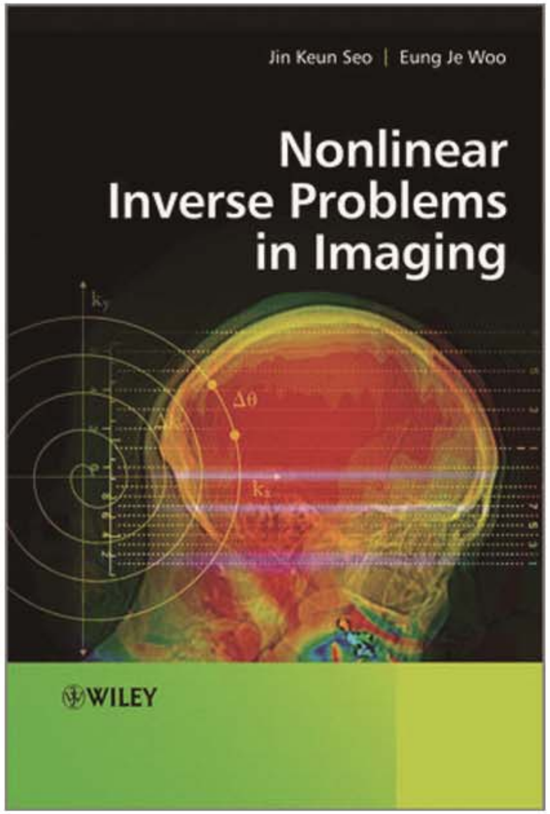
Partial Differential Equations



Part III-1: Energy functional & Sobolev Space

Homepage

http://www.seojinkeun.com/♯!pde-course-/c1laj

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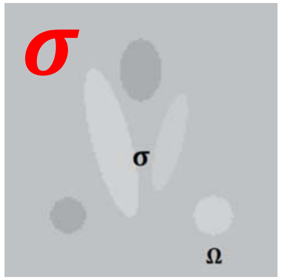
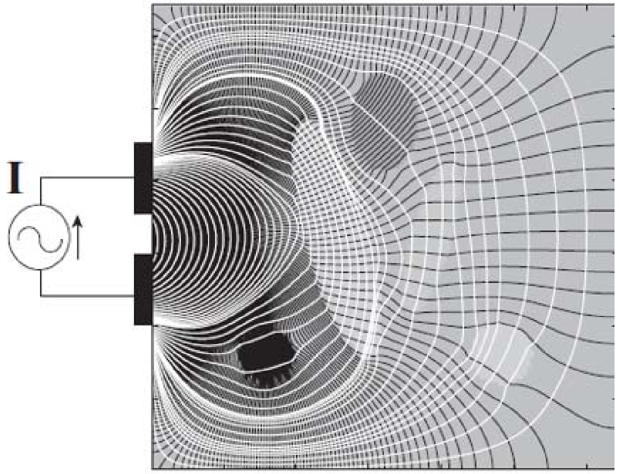
Elliptic PDE for inhomogeneous materials

$$\begin{cases} \nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega \\ u\big|_{\partial\Omega} = f \end{cases}$$



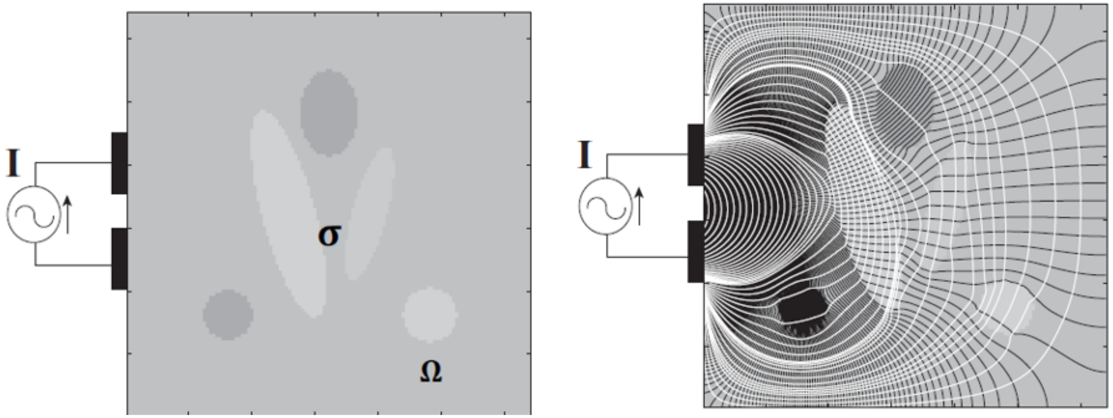


Note that σ can be discontinuous, i.e. $\sigma \notin C(\Omega)$.

- This PDE makes sense in classical sense when $\sigma \in C^1(\overline{\Omega})$ and $u \in C^2(\overline{\Omega})$.
- In practice, the material property σ may change abruptly. For example, a conductivity distribution σ inside the human body Ω may have a jump along the boundary of two different organs. Along such a boundary, the electrical field $\mathbf{E} = -\nabla u$ may be **discontinuous** due to interface conditions of the electric field (like the refractive condition of Snell's law). In this case, there exists no solution $u \in C^2(\Omega)$ in the classical sense.

How can we understand $\nabla \cdot (\sigma \nabla u) = 0$ when σ is discontinuous?

Answer: It can be understood by using variational framework.



Variational framework

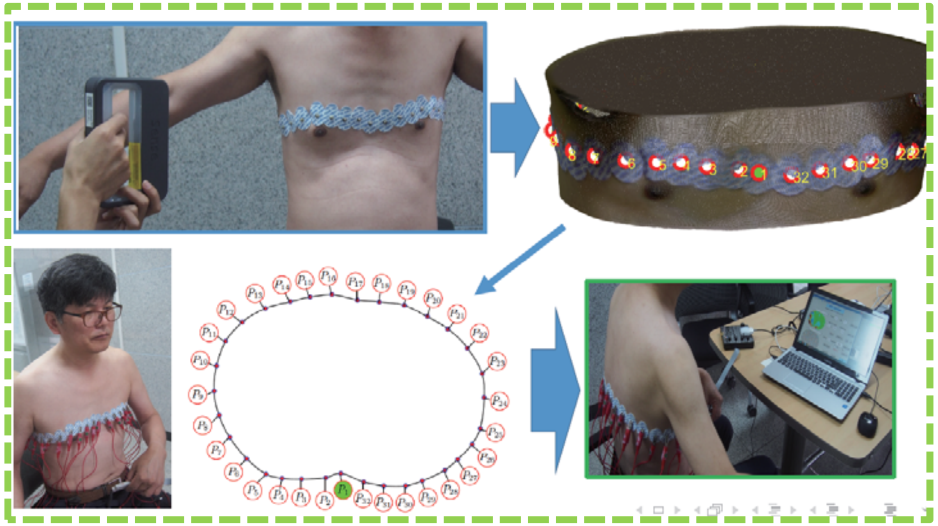
$$\boxed{\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega} \quad \Longleftrightarrow \quad \boxed{\int_\Omega \sigma \nabla u \cdot \nabla \phi = 0 \quad \forall \phi \in H_0^1(\Omega)}$$

Optional

Actual PDE problems are often very different from the textbook contents which deal with simplified toy models.

To find solutions that have practical significance and value, it is necessary to gain a deep understanding of the underlying physical phenomena with the parameter details of PDE models as well as the data acquisition systems.

- **How to get the domain Ω ?**
- **How to get the boundary data?**
- **What is the coefficient of elliptic PDE?**



응용에 실패한 (또는 실제문제 해결에 도움이 되지 않은) 대표적인 수학자 모델

제대로된 수학문제를 주세요. 그러면 풀어 드릴께요.

(무었이 어려운지 모르는 수학자)

u^j:=potential due to j-th injection current:

$$\begin{cases} \nabla\cdot(\gamma\nabla u^j)=0 & \text{in } \Omega \\ (u+z_k\gamma\frac{\partial u}{\partial \mathbf{n}})|_{\mathcal{E}_k}=U_k, & k=1,\cdots,E \\ \gamma\frac{\partial u^j}{\partial \mathbf{n}}=0 & \text{on } \partial\Omega\setminus\cup_{k=1}^{16}\mathcal{E}_k \\ \int_{\mathcal{E}_k}\gamma\frac{\partial u}{\partial \mathbf{n}}=0 & \text{if } k\neq j,j+1 \\ \int_{\mathcal{E}_j}\gamma\frac{\partial u^j}{\partial \mathbf{n}}\,ds=I=-\int_{\mathcal{E}_{j+1}}\gamma\frac{\partial u^j}{\partial \mathbf{n}}\,ds \end{cases}$$

I will not deal with this issue in this lecture.

Optional

What is the coefficient A (or σ) of elliptic PDE?

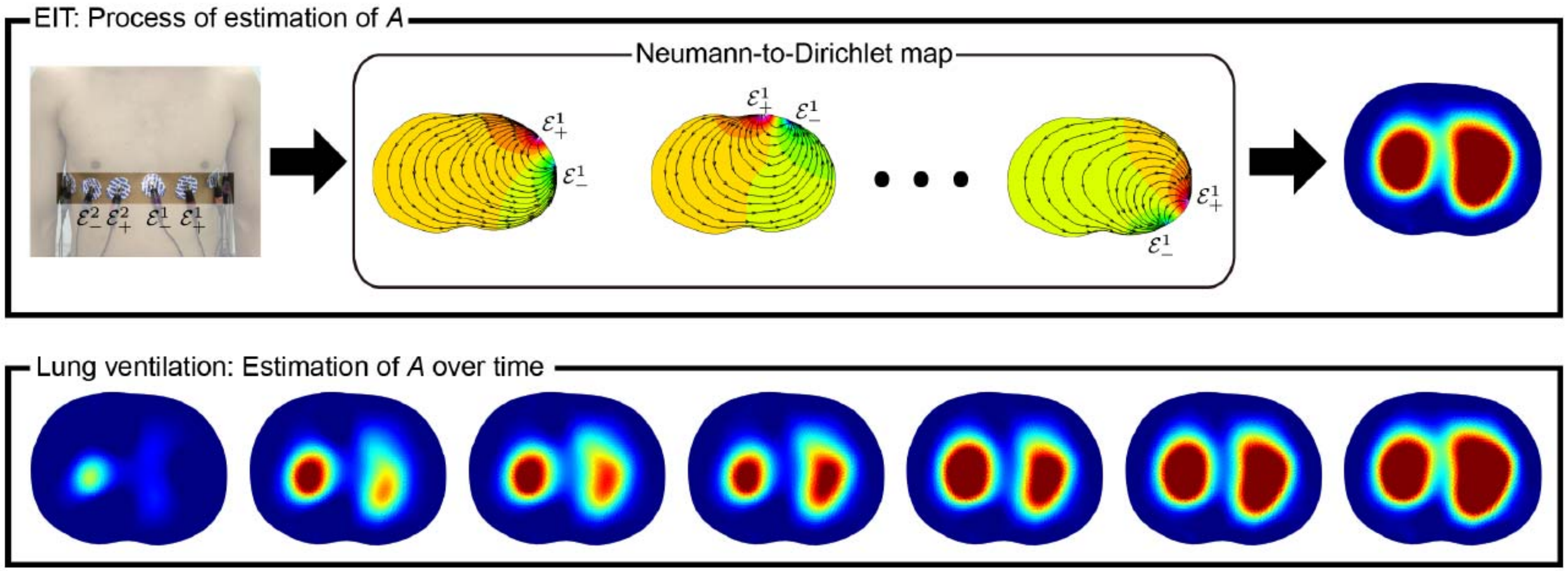
$$\nabla \cdot (A\nabla u) = 0$$

$$A = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$

The coefficient A (in electrical impedance area) should be understood as a **homogenized admittivity tensor** that depends on scale, position, cell structure including molecular compositions of cells, shape and direction of cells, cellular membranes, intra-and extra-cellular fluids, concentrations and mobilities of ions. **Homogenization and Harmonic analysis** (layer potential theories) are related to this subject.

(See the paper https://www.sciencedirect.com/science/article/pii/S0021782415001592 & effective conductivity in my webpage https://www.deepmediview.com/blank-15)

Under the assumption that $A = \sigma$ is isotropic, we get lung EIT(electrical impedance tomography) images.

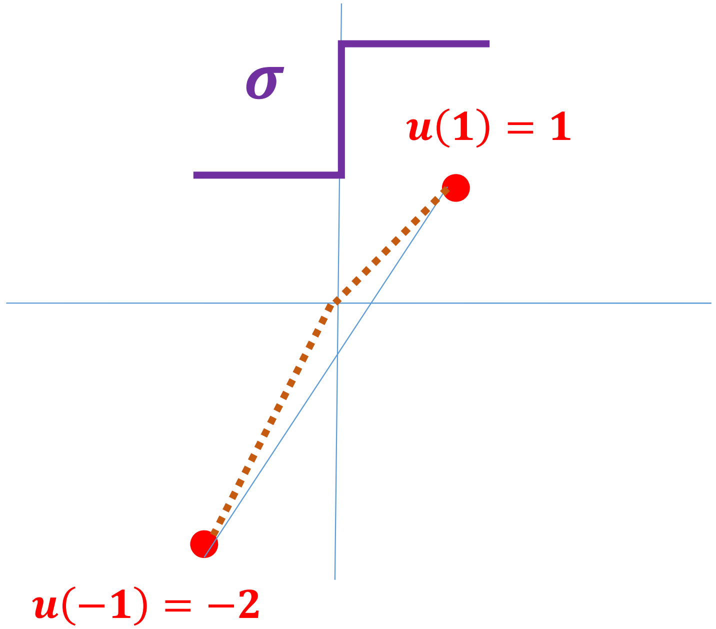


Understanding Variational Framework

$$\boxed{\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega} \iff \boxed{\int_\Omega \sigma \nabla u \cdot \nabla \phi = 0 \quad \forall \phi \in \mathbf{H}_0^1(\Omega)}$$

Consider the following one-dimensional Dirichlet problem:

$$\begin{cases} -\frac{d}{dx}\left(\sigma(x)\frac{d}{dx}u(x)\right) = 0 & \text{in } (-1,1) \\ u(-1) = -2, \quad u(1) = 1 \end{cases} \qquad \text{where } \sigma(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0. \end{cases}$$



- u satisfies $u''(x) = 0$ in $(-1,1) \setminus \{0\}$ with $u(-1) = 2, \ \ u(1) = 1$.
- $u \neq v$, the solution of $v''(x) = 0$ in $(-1,1)$ with $v(-1) = 2, \ \ v(1) = 1$.
- The classical derivative u' does not exist at $x = 0$:

$$u(x) = \begin{cases} x & \text{if } \ 0 \leq x < 1 \\ 2x & \text{if } \ -1 < x < 0 \end{cases} \qquad \& \qquad u'(x) = \begin{cases} 1 & \text{if } x > 0 \\ \nexists & \text{if } x = 0 \\ 2 & \text{if } x < 0 \end{cases}$$

- The difficulty can be removed by the use of the **variational framework**:

$$\int_{-1}^{1} \sigma(x) u'(x) \phi'(x) dx = 0 \qquad \forall \phi \in C_0^1(-1,1).$$

Why? See the next slide.

$$\frac{d}{dx}\left(\sigma\frac{d}{dx}u\right)=0 \quad in \ (-1,1) \iff \int_{-1}^{1}\sigma(x)u'(x)\phi'(x)dx=0,\ \forall\phi\in C_0^1([-1,1])$$

Transmission condition

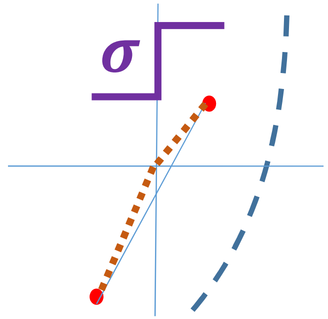
$$0=\int_{-1}^{1}\sigma(x)u'(x)\phi'(x)dx=\int_0^1 2u'(x)\phi'(x)dx+\int_{-1}^{0}u'(x)\phi'(x)dx=\boxed{(-2u'(0^+)+u'(0^-))}\phi(0)$$

$$-\int_0^1(2u'(x))'\phi(x)dx-2u'(0^+)\phi(0)$$

zero

$$-\int_{-1}^{0}(u'(x))'\phi(x)dx+u'(0^-)\phi(0)$$

zero



- A practically meaningful solution u should have a finite energy

$$\Phi(u):=\frac{1}{2}\int_{-1}^{1}\sigma|u'|^2dx<\infty.$$

- Indeed, $\boxed{u=\arg\min_{v\in\mathcal{A}}\Phi(v)}$, a minimizer of $\Phi(v)$ within the admissible set

$$\mathcal{A}:=\{v\ :\ \Phi(v)<\infty,\ \ v(-1)=-1,\ \ v(1)=2\}.$$

Elliptic PDE

$$\begin{cases} -\nabla \cdot (\sigma(\mathbf{r})\nabla u(\mathbf{r})) = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f \end{cases}$$

- The physically meaningful solution u must have a finite energy:

$$\Phi(v) = \int_\Omega \sigma(x)|\nabla v(\mathbf{r})|^2 dx < \infty.$$

- Hence, u should be contained in the set

$$\mathcal{A}_\Phi := \{v \in L^2(\Omega) : \Phi(v) < \infty\}.$$

- Assuming $0 < \inf_\Omega \sigma < \sup_\Omega \sigma < \infty$,

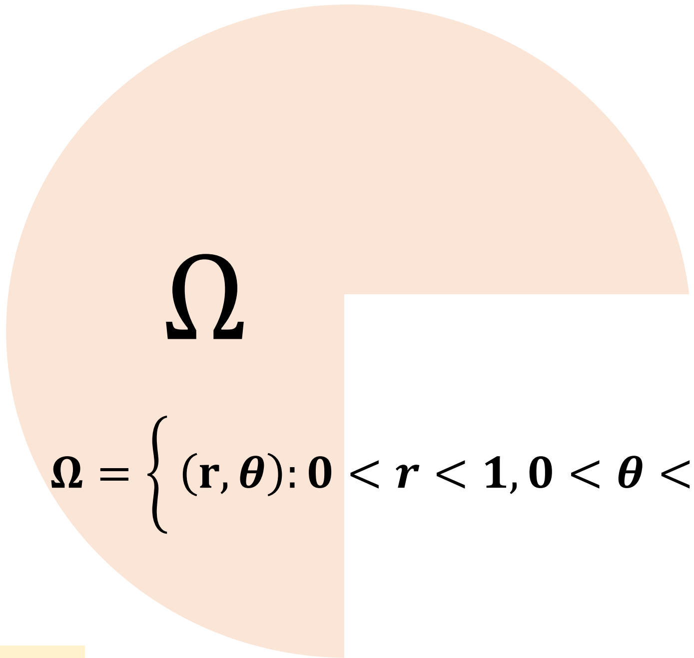
$$\mathcal{A}_\Phi = H^1(\Omega) := \{v : \|v\|_{H^1(\Omega)} := \sqrt{\int_\Omega |\nabla u|^2 + |u|^2 d\mathbf{x}} < \infty\}$$

This space is called the Sobolev space H^1.

Example The Dirichlet problem is known to be well-posed .

However, without the Sobolev constraint $H^1(\Omega)$, the Dirichlet problem can be **ill-posed.**

$$\begin{cases} \nabla \cdot \nabla \mathrm{u} = 0 \quad \text{in } \Omega \\ u\big|_{\partial\Omega} = 0 \end{cases}$$



$$\Omega = \left\{ (\mathrm{r}, \theta) : 0 < r < 1, 0 < \theta < \frac{3}{2}\pi \right\}$$

It is because there exist infinitely many solutions in $C^\infty(\Omega)$:

$$u(r,\theta) = \left(r^{\frac{2n}{3}} - r^{-\frac{2n}{3}}\right) \sin\frac{2n}{3}\theta, \qquad n = 1, 2, \cdots$$

$$\boxed{u \notin H^1(\Omega)}$$

Weak solution

Assume $\sigma \notin C(\bar{\Omega})$ and $0 < \sigma < \infty$. Consider

$$\begin{cases} -\nabla \cdot (\sigma(\mathbf{r})\nabla u(\mathbf{r})) = 0 \quad \text{in } \Omega \\ \qquad u|_{\partial\Omega} = f \end{cases} \qquad\qquad [\text{Assume } \sigma \notin C(\bar{\Omega})]$$

- When $u \in C^2(\bar{\Omega})$, there is no difference between the classical and variational problems.
- However, if $\sigma \notin C(\bar{\Omega})$, then the minimization problem has no solution in the class $C^2(\bar{\Omega})$. Obviously, the classical problem does not have a solution.
- We can construct a minimizing sequence $\{u_n\}$ in $C^2(\bar{\Omega})$ which is a Cauchy sequence with respect to the norm $\|u\|_{H^1(\Omega)}$.
- Although the Cauchy sequence $\{u_n\}$ does not converge within $C^2(\bar{\Omega})$, it converges in the Sobolev space $H^1(\Omega)$, the completion of $C^2(\bar{\Omega})$ with respect to the norm $\|u\|_{H^1(\Omega)}$. This means that we can solve the minimization and variational problem within the Sobolev space $H^1(\Omega)$.

Generalized derivative

The generalized derivative can be explained by means of the integrating by parts formula:

$$\int_\Omega u\,\partial_{x_i}\,\phi\,dx = -\int_\Omega \partial_{x_i}u\,\phi\,d\mathbf{x} \quad (\forall\phi\in C_0^\infty(\Omega)).$$

In general,

$$\int_\Omega u\,\partial^\alpha\,\phi\,d\mathbf{x} = (-1)^{|\alpha|}\int_\Omega \partial^\alpha u\,\phi\,d\mathbf{x} \quad (\forall\phi\in C_0^\infty(\Omega))$$

where the notions ∂^α and $|\alpha|$ are understood in the following way:

- $\alpha = (\alpha_1,\cdots,\alpha_n)\in\mathbb{N}_0^n,\ \mathbb{N}=\{1,2,\cdots\}$, and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$.
- $\partial^\alpha u = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}u$; for example, $\partial^{(2,0,3)}u = \partial_{x_1}^2\partial_{x_3}^3 u$.
- $|\alpha| = \sum_{k=1}^n \alpha_k$.

A function v_i satisfying the following equality behaves like the classical derivative $\partial_{x_i}u$:

$$\int_\Omega u\,\partial_{x_i}\phi\,dx = -\int_\Omega \underbrace{v_i}_{\partial_{x_i}u}\,\phi\,dx, \quad \forall\phi\in C_0^1(\Omega).$$

Sobolev space

We are now ready to introduce the Sobolev space $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ where Ω be a domain in $\mathbb{R}^n$ with its boundary $\partial\Omega$ and $1 \leq p < \infty$:

- $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ are the completion (or closure) of $C^\infty(\bar{\Omega})$ and $C_0^\infty(\Omega)$, respectively, with respect to the norm

$$\|u\|_{W^{1,p}} := \left(\int_\Omega |u|^p + |\nabla u|^p dx \right)^{1/p}.$$

  In other words,

$$W_0^{1,p}(\Omega) = \{u \ : \ \exists u_k \in C_0^\infty(\Omega) \ s.t. \lim_{k\to\infty} \|u_k - u\|_{W^{1,p}(\Omega)} = 0\}.$$

- $W^{2,p}(\Omega)$ and $W_0^{2,p}(\Omega)$ are the completion (or closure) of $C^\infty(\bar{\Omega})$ and $C_0^\infty(\Omega)$, respectively, with respect to the norm

$$\|u\|_{W^{2,p}} := \left(\int_\Omega |u|^p + |\nabla u|^p + |\nabla\nabla u|^p dx \right)^{1/p}.$$

- We denote $H^1(\Omega) = W^{1,2}(\Omega)$, $H_0^1(\Omega) = W_0^{1,2}(\Omega)$, $H^2(\Omega) = W^{2,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{2,2}(\Omega)$.

Optional

See my other lecture for details about "history of compactness, Sobolev space, and measure theory".

COMPACTNESS AND DIRICHLET'S PRINCIPLE

JIN KEUN SEO [1] AND HAMDI ZORGATI [2]

The concept of compactness and its introduction was highlighted by the famous debate between Riemann(1826-1866) and Weierstrass(1815-1897) regarding the convergence issue of the minimization problem in Dirichlet's principle. Riemann used the Dirichlet principle as follows: If u is the solution of Dirichlet's problem $\Delta u = 0$ in a bounded smooth domain $\Omega \subset \mathbb{R}^3$ with boundary data $u|_{\partial\Omega} = \phi \in C(\partial\Omega)$, then u can be obtained by the limit of the minimizing sequence $\{v_n\}$ of the energy functional

$$f(v) := \int_\Omega |\nabla v|^2 dx \tag{1.1}$$

within an admissible set such as $\mathcal{A} := \{v \in C^2(\Omega) \cap C(\overline{\Omega}) \ : \ v|_{\partial\Omega} = \phi\}$. In short, Riemann's observation was that f attains its minimum at a function u in the admissible class.

Theorem (Simplified version of Poincaré inequalities)

Let $\Omega = \{(x, y) \ : 0 < x, y < a\}$. A simplified version of the Poincaré inequality is

$$\sup_{u \in C_0^1(\Omega)} \frac{\|u\|_{L^2(\Omega)}}{\|\nabla u\|_{L^2(\Omega)}} \ \leq \ C$$

where C is a positive constant depending only on Ω.

The Poincaré inequality uses the special property of $u|_{\partial\Omega} = 0$ to get

$$|u(x, y)|^2 \leq \left| \int_0^a |\frac{\partial}{\partial x} u(x', y)| dx' \right|^2 \leq a \int_0^a |\nabla u(x', y)|^2 dx'$$

and hence

$$\int_\Omega |u|^2 \leq a^2 \int_\Omega |\nabla u|^2,$$

Theorem (Simplified version of trace inequalities)

Let $\Omega = \{(x, y) \ : 0 < x, y < a\}$. A simplified version of the trace inequality is

$$\sup_{u \in C^1(\overline{\Omega})} \frac{\|u\|_{L^2(\partial\Omega)}}{\|u\|_{H^1(\Omega)}} \ \leq \ C$$

where C is a positive constant depending only on Ω.

From the fundamental theorem of calculus,

$$\int_0^a |u(0, y)|^2 dy = \int_0^a \left| u(x, y) - \int_0^x \frac{\partial}{\partial x} u(x', y) dx' \right|^2 dy$$

and therefore

$$\begin{aligned}
\int_0^a |u(0, y)|^2 dy &\leq \int_0^a \left[|u(x, y)| + \int_0^a |\nabla u(x', y)| dx' \right]^2 dy \\
&\leq \int_0^a \left[2|u(x, y)|^2 + a \int_0^a |\nabla u(x', y)|^2 dx' \right] dy \\
&\leq 2 \int_0^a |u(x, y)|^2 dy + 2a \int_\Omega |\nabla u|^2 d\mathbf{r} \qquad (d\mathbf{r} = dxdy).
\end{aligned}$$

Optional

Theorem (Sobolev's inequality)

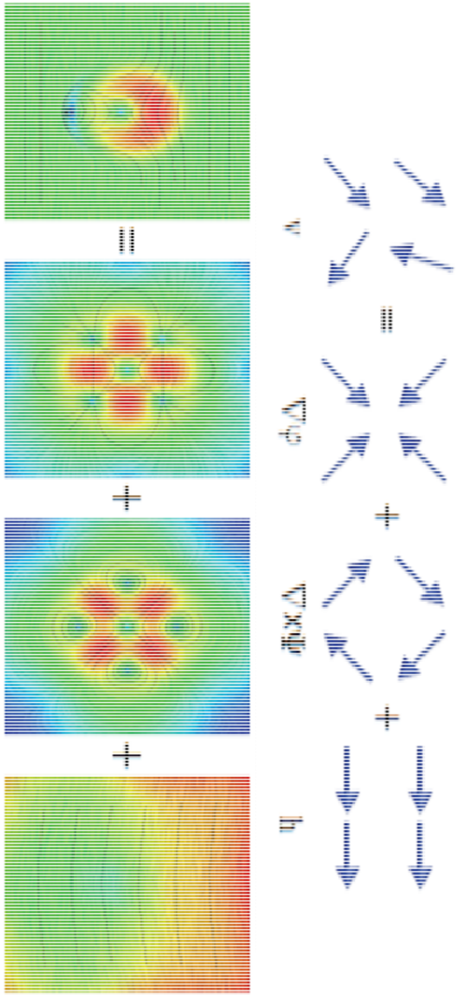
Let $u \in H^1(\mathbb{R}^n)$. The following inequality holds:

- *For $n \geq 3$,*
$$\|u\|^2_{L^{2n/(n-2)}} \leq C_n \|\nabla u\|^2_{L^2(\mathbb{R}^n)}$$
where $C_n = \frac{4}{n(n-2)} 2^{-2/n} \pi^{-1-1/n} [\Gamma(\frac{n+1}{2})]^{2/n}$.
- *For $n = 1$,* $\|u\|_{L^\infty(\mathbb{R})} \leq \frac{1}{2}\|u\|^2_{H^1(\mathbb{R})}$, $\quad \sup_{x,y} \frac{|u(x)-u(y)|}{|x-y|} \leq \|u'\|_{L^2}$.
- *For $n = 2$,* $\|u\|_{L^q(\mathbb{R}^2)} \leq C_q \|u\|^2_{H^1(\mathbb{R})}$, $\quad (\forall 2 \leq q < \infty)$ *where* $C_q \leq \left[q^{1-2/q}(q-1)^{-1+1/q}((q-2)/8\pi)^{1/2-1/q}\right]^2$.
- *Let Ω be a $C^{0,1}$-domain and $1 \leq p \leq q, m \geq 1$ and $k \leq m$. Then,*
$$\begin{aligned} \|u\|_{L^{np/(n-kp)}(\Omega)} &\leq C\|u\|_{W^{k,p}(\Omega)} && \text{if} \quad kp < n, \\ \|u\|_{C^m(\Omega)} &\leq C\,\|u\|_{W^{k+m,p}(\Omega)} && \text{if} \quad kp > n \end{aligned}$$
where C is independent of u.

For $n \geq 3$, the proof of the Sobolev inequality is based on the identity

$$u(\mathbf{x}) = \int_{\mathbb{R}^n} \nabla\left(\frac{c}{|\mathbf{x}-\mathbf{y}|^{n-2}}\right) \cdot \nabla u(\mathbf{y}) d\mathbf{y}.$$

where $c/|\mathbf{x}-\mathbf{y}|^{n-2}$ is the fundamental solution of Laplacian.



Helmholtz Decomposition

The Helmholtz decomposition states that any smooth vector field $\mathbf{F}$ in a smooth bounded domain Ω can be resolved into the sum of a divergence-free (solenoidal) vector field and a curl-free (irrotational) vector field.

Theorem

Every vector field $\mathbf{F}(\mathbf{r}) = (F_1(\mathbf{r}), F_2(\mathbf{r}), F_3(\mathbf{r})) \in [L^2(\Omega)]^3$ *can be decomposed into*

$$\mathbf{F}(\mathbf{r}) = -\nabla u(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r}) + \mathit{Harmonic} \quad \text{in } \Omega$$

where u *is a scalar function,* $\nabla \cdot \mathbf{A} = 0$ *and Harmonic is a vector field whose Laplacian is zero in* Ω*. Moreover,* u *and* $\mathbf{A}$ *are solutions of* $\nabla^2 u = \nabla \cdot \mathbf{F}$ *and* $\nabla^2 \mathbf{A} = \nabla \times \mathbf{F}$ *with appropriate boundary conditions. Hence, these can be uniquely determined up to harmonic functions;*

$$u(\mathbf{r}) = -\int_\Omega \frac{\nabla \cdot \mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' + \mathit{Harmonic}$$

and

$$\mathbf{A}(\mathbf{r}) = \int_\Omega \frac{\nabla \times \mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' + \mathit{Harmonic}.$$

Helmholtz's decomposition

$$\mathbf{F}(\mathbf{r}) = -\nabla \int_\Omega \frac{\nabla \cdot \mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' + \nabla \times \int_\Omega \frac{\nabla \times \mathbf{F}(\mathbf{r}')}{4\pi|\mathbf{r}-\mathbf{r}'|} d\mathbf{r}' + \mathit{Harmonic}$$

- Write

$$\mathbf{F}(\mathbf{r}) = \int_\Omega \delta(\mathbf{r}-\mathbf{r}')\mathbf{F}(\mathbf{r}')d\mathbf{r}' = -\int_\Omega \nabla^2 \left(\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\right) \mathbf{F}(\mathbf{r}')d\mathbf{r}'.$$

- Integrating by parts

$$\mathbf{F}(\mathbf{r}) = -\int_\Omega \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \nabla^2 \mathbf{F}(\mathbf{r}')d\mathbf{r}' + \mathit{Harmaonic}.$$

- Using the vector identity $-\nabla^2\mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) - \nabla(\nabla \cdot \mathbf{F})$,

$$\mathbf{F}(\mathbf{r}) = \int_\Omega \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \left[\nabla \times (\nabla \times \mathbf{F}) - \nabla(\nabla \cdot \mathbf{F})\right] dV + \mathit{Harmonic}, \qquad \mathbf{r} \in \Omega.$$

Integrating by parts again, we have the result.